

Variation of constants formula and exponential dichotomy for non autonomous non densely defined Cauchy problems

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Abstract: In this paper we prove a variation of constants formula for a non autonomous and non homogeneous Cauchy problems whenever the linear part is not densely defined and is not a Hille-Yosida operator. By using this variation of constants formula we derive a necessary and sufficient conditions for the existence of exponential dichotomy for the evolution family generated by the associated non autonomous homogeneous problem. We also prove a persistence result of the exponential dichotomy for small perturbations. Finally we illustrate our result by consider a parabolic equation with non local and non autonomous boundary conditions.

Keywords : Non autonomous Cauchy problem, non densely defined Cauchy problem, exponential dichotomy.

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1 Introduction

In this article we consider the following non homogeneous non autonomous problem

$$\frac{du(t)}{dt} = (A + B(t))u(t) + f(t), \text{ for } t \geq t_0, \text{ and } u(t_0) = x \in \overline{D(A)}, \quad (1.1)$$

where $t_0 \in \mathbb{R}$, $A : D(A) \subset X \rightarrow X$ is a linear operator (possibly with non dense domain i.e $\overline{D(A)} \subsetneq X$) on a Banach space $(X, \|\cdot\|)$, $\{B(t)\}_{t \in \mathbb{R}} \subset \mathcal{L}(\overline{D(A)}, X)$ is a locally bounded and strongly continuous family of bounded linear operators and $f \in L^1_{loc}(\mathbb{R}, X)$.

Throughout this article, we will make the following assumption on the linear operator $(A, D(A))$.

Assumption 1.1 *We assume that*

i) *There exist two constants $\omega \in \mathbb{R}$ and $M \geq 1$, such that $(\omega, +\infty) \subset \rho(A)$ and*

$$\|(\lambda I - A)^{-k}\|_{\mathcal{L}(\overline{D(A)})} \leq M (\lambda - \omega)^{-k}, \quad \forall \lambda > \omega, k \geq 1.$$

ii) $\lim_{\lambda \rightarrow +\infty} (\lambda I - A)^{-1}x = 0, \quad \forall x \in X$.

Recall that A is a Hille-Yosida operator if there exist two constants $\omega \in \mathbb{R}$ and $M \geq 1$, such that $(\omega, +\infty) \subset \rho(A)$ and

$$\|(\lambda I - A)^{-k}\|_{\mathcal{L}(X)} \leq M (\lambda - \omega)^{-k}, \quad \forall \lambda > \omega, \forall k \geq 1.$$

In this article, we will not assume that A is a Hille-Yosida operator since in Assumption 1.1-i) the operator norm is taken into $\overline{D(A)}$ instead of X . Set

$$X_0 := \overline{D(A)}$$

and denote by A_0 the part of A on X_0 that is

$$A_0x = Ax, \quad \forall x \in D(A_0)$$

and

$$D(A_0) := \{x \in D(A) : Ax \in X_0\}.$$

Then it has been proved (see [31, Lemma 2.1 and Lemma 2.2]) Assumption 1.1 is equivalent to $\rho(A) \neq \emptyset$ and A_0 is a Hille-Yosida linear operator on X_0 . Therefore A_0 generates a strongly continuous semigroup $\{T_{A_0}(t)\}_{t \geq 0} \subset \mathcal{L}(X_0)$. An important and useful approach to investigate such a non-densely defined Cauchy problems is to use the integrated semigroup theory, which was first introduced by Arendt [3, 4]. The integrated semigroup generated by A , namely $\{S_A(t)\}_{t \geq 0}$ is a strongly continuous family of bounded linear operator on X , which commute with the resolvent of A , and such that for each $x \in X$ the map $t \rightarrow S_A(t)x$ is an integrated solution of the Cauchy problem

$$\frac{du}{dt} = Au(t) + x, \text{ for } t \geq 0 \text{ and } u(0) = 0. \quad (1.2)$$

Considering the Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t), \text{ for } t \geq 0 \text{ and } u(0) = 0, \quad (1.3)$$

with $f \in L^1((0, \tau), X)$. When A is a Hille-Yosida operator, it was proved by Kellermann and Hieber [24] that

$$t \rightarrow (S_A * f)(t) := \int_0^t S_A(t-s)f(s)ds$$

is continuously differential and the derivative

$$u(t) := \frac{d}{dt} (S_A * f)(t)$$

is an integrated (or mild) solution of (1.3). That is to say

$$\int_0^t u(s)ds \in D(A), \forall t \geq 0,$$

and

$$u(t) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \forall t \geq 0.$$

The uniqueness of mild solution has been proved by Thieme [38].

The next assumption needed to use perturbations $f(t)$ that are continuous in time is the following.

Assumption 1.2 *For each $\tau > 0$ and each $f \in C([0, \tau], X)$ we assume that there exists $u_f \in C([0, \tau], X_0)$ an integrated (or mild) solution of*

$$\frac{du_f}{dt} = Au_f(t) + f(t), \text{ for } t \geq 0 \text{ and } u_f(0) = 0.$$

Moreover we assume that there exists a non decreasing map $\delta : [0, +\infty) \rightarrow [0, +\infty)$ such that

$$\|u_f(t)\| \leq \delta(t) \sup_{s \in [0, t]} \|f(s)\|, \quad \forall t \geq 0, \quad (1.4)$$

with

$$\delta(t) \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

Let $f \in C([0, +\infty), X)$ be fixed. The existence of mild solution in Assumption 1.2 is equivalent to the continuous time differentiability of the map $t \rightarrow (S_A * f)(t)$ from $[0, +\infty)$ into X . Moreover

$$u_f(t) = \frac{d}{dt} (S_A * f)(t), \forall t \geq 0,$$

whenever the mild solution exists. Define

$$(S_A \diamond f)(t) := \frac{d}{dt} (S_A * f)(t), \forall t \geq 0.$$

The foregoing Assumption 1.2 needs justification. In fact if A is Hille-Yosida operator, then Assumption 1.2 holds true as long as $t \rightarrow f(t)$ is continuous (see Kellermann and Hieber [24]) and we have the following estimate

$$\|(S_A \diamond f)(t)\| \leq M \int_0^t e^{\omega(t-s)} \|f(s)\| ds$$

therefore

$$\|(S_A \diamond f)(t)\| \leq \left(M \int_0^t e^{\omega s} ds \right) \sup_{s \in [0, t]} \|f(s)\|, \quad \forall t \geq 0, \quad (1.5)$$

which clearly show that (1.4) is a generalization of (1.5). Hence since we do not assume that A is Hille-Yosida, Assumption 1.2 becomes an issue in order to obtain the existence of integrated solutions for the non-homogeneous equation (1.1). This question has been studied by Magal and Ruan [29] and by Thieme [40] in the general case, and by Ducrot, Magal and Prevost [15] in the almost sectorial case. Assumptions 1.1 and 1.2 are justified by the fact that in several situations the linear operator $(A, D(A))$ is not Hille-Yosida while the differentiability of $t \rightarrow (S_A * f)(t)$ and the estimate (1.4) can be obtained if $(A_0, D(A_0))$ is a Hille-Yosida operator.

The following assumption will be required in order to deal with the existence of integrated solutions for the non homogeneous equation (1.1).

Assumption 1.3 *Let $\{B(t)\}_{t \in \mathbb{R}} \subset \mathcal{L}(X_0, X)$ be a family of bounded linear operators. We assume that $t \rightarrow B(t)$ is strongly continuous from \mathbb{R} into $\mathcal{L}(X_0, X)$, that is to say that for each $x \in X_0$ the map $t \rightarrow B(t)x$ is continuous from \mathbb{R} into X . We assume that there exists $b \in C(\mathbb{R}, \mathbb{R}_+)$ such that*

$$\|B(t)\|_{\mathcal{L}(X_0, X)} \leq b(t), \quad \forall t \in \mathbb{R}.$$

The foregoing assumptions will allows us to obtain the existence of an *evolution family* (see Definition 1.4 below) for the homogeneous Cauchy problem (1.6). Before proceeding let us introduce the notation

$$\Delta := \{(t, s) \in \mathbb{R}^2 : t \geq s\},$$

and recall the notion of evolution family.

Definition 1.4 *Let $(Z, \|\cdot\|)$ be a Banach space. A two parameters family of bounded linear operators on Z , $\{U(t, s)\}_{(t, s) \in \Delta}$ is an evolution family if*

i) *For each $t, r, s \in \mathbb{R}$ with $t \geq r \geq s$*

$$U(t, t) = I_{\mathcal{L}(Z)} \quad \text{and} \quad U(t, r)U(r, s) = U(t, s).$$

ii) *For each $x \in Z$, the map $(t, s) \rightarrow U(t, s)x$ is continuous from Δ into Z .*

If in addition there exist two constants $\widehat{M} \geq 1$ and $\widehat{\omega} \in \mathbb{R}$ such that

$$\|U(t, s)\|_{\mathcal{L}(Z)} \leq \widehat{M} e^{\widehat{\omega}(t-s)}, \quad \forall (t, s) \in \Delta,$$

we say that $\{U(t, s)\}_{(t, s) \in \Delta}$ is an exponentially bounded evolution family.

Consider the following homogeneous equation for each $t_0 \in \mathbb{R}$

$$\frac{du(t)}{dt} = (A + B(t))u(t), \text{ for } t \geq t_0 \text{ and } u(t_0) = x \in X_0. \quad (1.6)$$

By using [29, Theorem 5.2] and [30, Proposition 4.1] we obtain the following Proposition.

Proposition 1.5 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then the homogeneous Cauchy problem (1.6) generates a unique evolution family $\{U_B(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(X_0)$. Moreover $U_B(\cdot, t_0)x_0 \in C([t_0, +\infty), X_0)$ is the unique solution of the fixed point problem*

$$U_B(t, t_0)x_0 = T_{A_0}(t - t_0)x_0 + \frac{d}{dt} \int_{t_0}^t S_A(t - s)B(s)U_B(s, t_0)x_0 ds, \quad \forall t \geq t_0. \quad (1.7)$$

If we assume in addition that

$$\sup_{t \in \mathbb{R}} b(t) < +\infty$$

then the evolution family $\{U_B(t, s)\}_{(t,s) \in \Delta}$ is exponentially bounded.

The following theorem provides an approximation formula of the solutions of equation (1.1). This is the first main result.

Theorem 1.6 (Variation of constants formula) *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then for each $t_0 \in \mathbb{R}$, each $x_0 \in X_0$ and each $f \in C([t_0, +\infty], X)$ the unique integrated solution $u_f \in C([t_0, +\infty], X_0)$ of (1.1) is given by*

$$u_f(t) = U_B(t, t_0)x_0 + \lim_{\lambda \rightarrow +\infty} \int_{t_0}^t U_B(t, s)\lambda R_\lambda(A)f(s)ds, \quad \forall t \geq t_0, \quad (1.8)$$

where the limit exists in X_0 . Moreover the convergence in (1.8) is uniform with respect to $t, t_0 \in I$ for each compact interval $I \subset \mathbb{R}$.

Our second main result deal with a necessary and sufficient condition for the evolution family (generated by the homogeneous problem associated to system (1.1)) to have an exponential dichotomy. To be more precise let us first recall some definitions and state our result.

Definition 1.7 *Let $(Z, \|\cdot\|_Z)$ be a Banach space. We say that $\{\Pi(t)\}_{t \in \mathbb{R}} \subset \mathcal{L}(Z)$ is strongly continuous family of projectors on Z if*

$$\Pi(t)\Pi(t) = \Pi(t), \quad \forall t \in \mathbb{R},$$

and for each $x \in Z$, $t \rightarrow \Pi(t)x$ is continuous from \mathbb{R} into Z .

The following notion of exponential dichotomy will be used in this paper. We refer for instance to [16, 17, 20, 21, 22, 27] and the references therein.

Definition 1.8 *Let $(Z, \|\cdot\|_Z)$ be a Banach space. We say that an evolution family $\{U(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(Z)$ has an exponential dichotomy with constant $\kappa \geq 1$ and exponent $\beta > 0$ if and only if the following properties are satisfied*

- i) There exist two strongly continuous family of projectors $\{\Pi^+(t)\}_{t \in \mathbb{R}}$ and $\{\Pi^-(t)\}_{t \in \mathbb{R}}$ on Z such that

$$\Pi^+(t) + \Pi^-(t) = I_{\mathcal{L}(Z)}, \quad \forall t \in \mathbb{R}.$$

Then we define for all $t \geq s$

$$U^+(t, s) := U(t, s)\Pi^+(s) \quad \text{and} \quad U^-(t, s) := U(t, s)\Pi^-(s).$$

- ii) For all $(t, s) \in \Delta$ we have $\Pi^+(t)U(t, s) = U(t, s)\Pi^+(s)$ and then $\Pi^-(t)U(t, s) = U(t, s)\Pi^-(s)$.
- iii) For all $(t, s) \in \Delta$ the restricted linear operator $U(t, s)\Pi^-(s)$ is invertible from $\Pi^-(s)(Z)$ into $\Pi^-(t)(Z)$ with inverse denoted by $\bar{U}^-(s, t)$ and we set

$$U^-(s, t) := \bar{U}^-(s, t)\Pi^-(t).$$

- iv) For all $(t, s) \in \Delta$

$$\|U^+(t, s)\|_{\mathcal{L}(Z)} \leq \kappa e^{-\beta(t-s)} \quad \text{and} \quad \|U^-(s, t)\|_{\mathcal{L}(Z)} \leq \kappa e^{-\beta(t-s)}.$$

In the foregoing Definition 1.8 the notation $+$ and $-$ are used to refer respectively the forward time and the backward time.

Definition 1.9 Let $f \in L^1_{loc}(\mathbb{R}, X)$ be fixed. A function $u \in C(\mathbb{R}, X_0)$ is an integrated (or mild) solution of (1.1) if and only if for each $t \geq t_0$

$$\int_{t_0}^t u(r)dr \in D(A)$$

and

$$u(t) = x + A \int_{t_0}^t u(r)dr + \int_{t_0}^t [B(r)u(r) + f(r)]dr.$$

Then our second main result split into the following two theorems.

Theorem 1.10 Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Then the following assertions are equivalent

- i) The evolution family $\{U_B(t, s)\}_{(t, s) \in \Delta}$ has an exponential dichotomy.
- ii) For each $f \in BC(\mathbb{R}, X)$, there exists a unique integrated solution $u \in BC(\mathbb{R}, X_0)$ of (1.1).

Theorem 1.11 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

If U_B has an exponential dichotomy with exponent $\beta > 0$, then for each $\eta \in [0, \beta)$ and each $f \in BC^\eta(\mathbb{R}, X)$ with

$$BC^\eta(\mathbb{R}, X) := \left\{ f \in C(\mathbb{R}, Z) : \|f\|_\eta := \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|f(t)\|_Z < +\infty \right\}$$

there exists a unique integrated solution $u \in BC^\eta(\mathbb{R}, X_0)$ of (1.1) which is given by

$$u_f(t) = \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t U_B^+(t, s) \lambda R_\lambda(A) f(s) ds - \int_t^{+\infty} U_B^-(t, s) \lambda R_\lambda(A) f(s) ds, \quad \forall t \in \mathbb{R}. \quad (1.9)$$

Moreover the following properties hold true

- i) The limit (1.9) exists uniformly on compact subset of \mathbb{R} .*
- ii) If f is bounded and uniformly continuous with relatively compact range then the limit (1.9) is uniform on \mathbb{R} .*
- iii) For each $\nu \in (-\beta, 0)$ there exists $C(\nu, \kappa, \beta) > 0$ such that*

$$\|u_f\|_\eta \leq C(\nu, \kappa, \beta) \|f\|_\eta, \quad \forall \eta \in [0, -\nu].$$

Let us mention that similar results have been investigated in [20] where $(A, D(A))$ is a Hille-Yosida linear operator in the context of extrapolated semigroups. However the fact that $(A, D(A))$ is not Hille-Yosida induces several difficulties that need new technical arguments in order to prove our more general results that by using integrated semigroups theory.

The paper is organized as follows. In section 2 we recall some results concerning integrated semigroups and define the notion of integrated solution for system (1.1). Section 3 is devoted to the proof of Theorem 1.6 concerning the variation of constants formula. In Section 4 we prove some uniform convergence results. Finally Theorems 1.10 and 1.11 are proved in Section 5.

2 Preliminaries

In the subsequent lemma we summarize some results proved in Magal and Ruan [31, Lemma 2.1 and Lemma 2.2]. For more simplicity in the notations we define

$$R_\lambda(A) := (\lambda I - A)^{-1}, \quad \forall \lambda > \omega.$$

Lemma 2.1 *Let Assumption 1.1 be satisfied. Then we have*

$$\rho(A) = \rho(A_0).$$

Moreover, we have the following properties

i) For each $\lambda > \omega$

$$D(A_0) = R_\lambda(A)(X_0) \quad \text{and} \quad R_\lambda(A)|_{X_0} = R_\lambda(A_0).$$

ii) $\overline{D(A_0)} = X_0$.

iii) $\lim_{\lambda \rightarrow +\infty} R_\lambda(A)x = x, \quad \forall x \in X_0$.

Remark 2.2 It can be easily proved that $\lim_{\lambda \rightarrow +\infty} R_\lambda(A)x = x$ uniformly for x in a relatively compact subset of X_0 . This property will be often used in this paper.

Note that if $(A, D(A))$ satisfies Assumption 1.1 then by Lemma 2.1 we have

$$\|R_\lambda(A_0)^k\|_{\mathcal{L}(X_0)} \leq M(\lambda - \omega)^{-k}, \quad \forall \lambda > \omega, \quad k \geq 1 \quad \text{and} \quad \overline{D(A_0)} = X_0.$$

Therefore $(A_0, D(A_0))$ generates a strongly continuous semigroup $\{T_{A_0}(t)\}_{t \geq 0} \subset \mathcal{L}(X_0)$ with

$$\|T_{A_0}(t)\|_{\mathcal{L}(X_0)} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

The characterization of an integrated semigroup is summarized in the definition below.

Definition 2.3 Let $(X, \|\cdot\|)$ be a Banach space. A family of bounded linear operators $\{S(t)\}_{t \geq 0}$ on X is called an integrated semigroup if

i) $S(0)x = 0, \forall x \in X$.

ii) $t \rightarrow S(t)x$ is continuous on $[0, +\infty)$ for each $x \in X$.

iii) For each $t \geq 0$, $S(t)$ satisfies

$$S(s)S(t) = \int_0^s [S(r+t) - S(r)]dr, \quad \forall s \geq 0.$$

The integrated semigroup $\{S(t)\}_{t \geq 0}$ is said non-degenerate if

$$S(t)x = 0, \quad \forall t \geq 0 \Rightarrow x = 0.$$

Moreover we will say that $(A, D(A))$ generates an integrated semigroup $\{S_A(t)\}_{t \geq 0} \subset \mathcal{L}(X, X_0)$ that is

$$x \in D(A) \quad \text{and} \quad y = Ax \Leftrightarrow S_A(t)x = tx + \int_0^t S(s)yds, \quad \forall t \geq 0.$$

The following result is well known in the context of integrated semigroups.

Proposition 2.4 Let Assumption 1.1 be satisfied. Then $(A, D(A))$ generates a uniquely determined non-degenerate exponentially bounded integrated semigroup with

$$\|S_A(t)\|_{\mathcal{L}(X)} \leq \hat{M}e^{\hat{\omega}t},$$

where $\hat{M} > 0$, $\hat{\omega} > 0$ and $(\hat{\omega}, +\infty) \in \rho(A)$.

Moreover the following properties hold

i) For each $x \in X$, each $t \geq 0$, each $\mu > \omega$, $S_A(t)x$ is given by

$$S_A(t)x = (\lambda I - A) \int_0^t T_{A_0}(s) ds (\lambda I - A)^{-1}, \quad \forall \lambda > \omega,$$

or equivalently

$$S_A(t)x = \mu \int_0^t T_{A_0}(s) R_\mu(A) x ds + [I - T_{A_0}(t)] R_\mu(A) x.$$

ii) The map $t \rightarrow S_A(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{dS_A(t)x}{dt} = T_{A_0}(t)x, \quad \forall t \geq 0, \quad \forall x \in X_0.$$

Next we give the notion of integrated solution for system (1.1).

Definition 2.5 Let $t_0 \in \mathbb{R}$ and let $f \in L^1_{loc}((t_0, +\infty), X)$ be fixed. A function $u \in C([t_0, +\infty), X_0)$ is an integrated (or mild) solution of (1.1) if and only if for each $t \geq t_0$

$$\int_{t_0}^t u(r) dr \in D(A)$$

and

$$u(t) = x + A \int_{t_0}^t u(r) dr + \int_{t_0}^t [B(r)u(r) + f(r)] dr.$$

The following result is a direct consequence of Theorem 2.10 in [30].

Theorem 2.6 Let Assumptions 1.1 and 1.2 be satisfied. Let $t_0 \in \mathbb{R}$ be fixed. Then for all $f \in C([t_0, +\infty), X)$, the map $t \rightarrow (S_A * f(t_0 + \cdot))(t - t_0)$ is continuously differentiable from $[t_0, +\infty)$ into X and satisfies the following properties

i) $(S_A * f(t_0 + \cdot))(t - t_0) \in D(A)$, $\forall t \geq t_0$.

ii) If we set

$$u(t) := (S_A \diamond f(t_0 + \cdot))(t - t_0), \quad \forall t \geq t_0,$$

then the following hold

$$u(t) = A \int_{t_0}^t u(s) ds + \int_{t_0}^t f(s) ds, \quad \forall t \geq t_0,$$

and

$$\|u(t)\| \leq \delta(t - t_0) \sup_{s \in [t_0, t]} \|f(s)\|, \quad \forall t \geq t_0.$$

iii) For all $\lambda \in (\omega, +\infty)$ we have for each $t \geq t_0$

$$R_\lambda(A) \frac{d}{dt} (S_A * f(t_0 + \cdot))(t - t_0) = \int_{t_0}^t T_{A_0}(t - s) R_\lambda(A) f(s) ds.$$

As a consequence of *iii*) in Theorem 2.6, we obtain the following approximation formula

$$\frac{d}{dt} \int_{t_0}^t S_A(t-s)f(s)ds = \lim_{\lambda \rightarrow +\infty} \int_{t_0}^t T_{A_0}(t-s)\lambda R_\lambda(A)f(s)ds, \quad \forall t \geq t_0. \quad (2.1)$$

It also follows that for each $t, h \geq 0$

$$(S_A \diamond f)(t+h) = T_{A_0}(h)(S_A \diamond f)(t) + (S_A \diamond f(t+\cdot))(h). \quad (2.2)$$

As an immediate consequence of Theorem 2.6 we obtain the following lemma.

Lemma 2.7 *Let Assumptions 1.1 and 1.2 be satisfied. Let $f \in C(\mathbb{R}, X)$. Then the map $(t, t_0) \rightarrow (S_A \diamond f(t_0 + \cdot))(t - t_0)$ is continuous from Δ into X .*

Proof. Let $(t, t_0), (s, s_0) \in \Delta$. We have

$$\begin{aligned} I &:= (S_A \diamond f(t_0 + \cdot))(t - t_0) - (S_A \diamond f(s_0 + \cdot))(s - s_0) \\ &= (S_A \diamond [f(t_0 + \cdot) - f(s_0 + \cdot)])(t - t_0) \\ &\quad + (S_A \diamond f(s_0 + \cdot))(t - t_0) - (S_A \diamond f(s_0 + \cdot))(s - s_0) \end{aligned}$$

hence by using (2.2)

$$\begin{aligned} I &= (S_A \diamond [f(t_0 + \cdot) - f(s_0 + \cdot)])(t - t_0) \\ &\quad + [T_{A_0}((t - t_0) - (s - s_0)) - I](S_A \diamond f(s_0 + \cdot))(s - s_0) \\ &\quad + (S_A \diamond f(s_0 + (s - s_0) + \cdot))((t - t_0) - (s - s_0)) \end{aligned}$$

whenever $t - t_0 \geq s - s_0$. The result follows by using the uniform continuity of f on bounded intervals. ■

By using [30, Proposition 4.1] we obtain the following lemma.

Lemma 2.8 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Let $t_0 \in \mathbb{R}$ be fixed. Then for each $x_0 \in X_0$ and $f \in C([t_0, +\infty), X)$ there exists a unique integrated solution $u_f \in C([t_0, +\infty), X_0)$ of (1.1) given by*

$$u_f(t) = T_{A_0}(t - t_0)x_0 + \frac{d}{dt}(S_A * ((Bu_f)(t_0 + \cdot) + f(t_0 + \cdot))(t - t_0)), \quad \forall t \geq t_0,$$

or equivalently

$$u_f(t) = T_{A_0}(t - t_0)x_0 + (S_A \diamond ((Bu_f)(t_0 + \cdot) + f(t_0 + \cdot))(t - t_0)), \quad \forall t \geq t_0,$$

where we have used the notation $(Bu_f)(t) := B(t)u_f(t)$ for every $t \geq t_0$.

The next result is due to Magal and Ruan [30, Proposition 2.14] and is one of the main tools in studying integrated solution for non Hille-Yosida operators. It reads as

Proposition 2.9 *Let Assumption 1.1 be satisfied. Let $\varepsilon > 0$ be given and fixed. Then, for each $\tau_\varepsilon > 0$ satisfying $M\delta(\tau_\varepsilon) \leq \varepsilon$, we have*

$$\left\| \frac{d}{dt}(S_A * f)(t) \right\| \leq C(\varepsilon, \gamma) \sup_{s \in [0, t]} e^{\gamma(t-s)} \|f(s)\|, \quad \forall t \geq 0,$$

whenever $\gamma \in (\omega, +\infty)$, $f \in C(\mathbb{R}_+, X)$ with

$$C(\varepsilon, \gamma) := \frac{2\varepsilon \max(1, e^{-\gamma\tau_\varepsilon})}{1 - e^{(\omega-\gamma)\tau_\varepsilon}}.$$

3 A variation of constants formula

In this section we will prove the first main result of this paper. It deal with the representation of the integrated solution of (1.1) in term of the evolution family $\{U_B(t, s)\}_{(t,s) \in \Delta}$. This result generalize [20, Theorem 2.2] to the context of non Hille-Yosida operator. The proof will be given by using several technical lemmas. Note that a direct consequence of Theorem 1.6 is the following

Corollary 3.1 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then for each $t_0 \in \mathbb{R}$, each $x_0 \in X_0$ and each $f \in C([t_0, +\infty], X_0)$ the unique integrated solution $u_f \in C([t_0, +\infty], X_0)$ of (1.1) is given by*

$$u_f(t) = U_B(t, t_0)x_0 + \int_{t_0}^t U_B(t, s)f(s)ds, \quad \forall t \geq t_0.$$

Next we prove some technical lemmas that will be crucial for the proof of Theorem 1.6.

Lemma 3.2 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Then for each $h \in C(\Delta, X)$ the following equality holds*

$$\begin{aligned} \int_{t_0}^t \frac{d}{dt} \left[\int_s^t S_A(t-r)h(r, s)dr \right] ds = \\ \frac{d}{dt} \int_{t_0}^t S_A(t-r) \left[\int_{t_0}^r h(r, s)ds \right] dr, \end{aligned}$$

for all $(t, t_0) \in \Delta$.

Proof. Let $t_0 \in \mathbb{R}$ be fixed. Let $s \geq t_0$ be given. Then observing that $h(\cdot, s) \in C([s, +\infty), X)$ one can apply Theorem 2.6 to obtain for all $t \geq s$ and $\lambda > \omega$

$$\int_s^t T_{A_0}(t-r)\lambda R_\lambda(A)h(r, s)dr = \lambda R_\lambda(A) \frac{d}{dt} \int_s^t S_A(t-r)h(r, s)dr. \quad (3.1)$$

Thus integrating the both sides of (3.1) and using Fubini's theorem we obtain for each $t \geq t_0$ and $\lambda > \omega$

$$\begin{aligned} \lambda R_\lambda(A) \int_{t_0}^t \left[\frac{d}{dt} \int_s^t S_A(t-r)h(r, s)dr \right] ds &= \int_{t_0}^t \left[\int_s^t T_{A_0}(t-r)\lambda R_\lambda(A)h(r, s)dr \right] ds \\ &= \int_{t_0}^t \left[\int_{t_0}^r T_{A_0}(t-r)\lambda R_\lambda(A)h(r, s)ds \right] dr \\ &= \int_{t_0}^t T_{A_0}(t-r)\lambda R_\lambda(A) \left[\int_{t_0}^r h(r, s)ds \right] dr. \end{aligned}$$

Now observing that

$$\int_{t_0}^t \left[\frac{d}{dt} \int_s^t S_A(t-r)h(r, s)dr \right] ds \in X_0, \quad \forall t \geq t_0,$$

the result follows since we have

$$\lim_{\lambda \rightarrow +\infty} \lambda R_\lambda(A) \int_{t_0}^t \left[\frac{d}{dt} \int_s^t S_A(t-r) h(r, s) dr \right] ds = \int_{t_0}^t \left[\frac{d}{dt} \int_s^t S_A(t-r) h(r, s) dr \right] ds,$$

for all $t \geq t_0$ and (see equality (2.1))

$$\lim_{\lambda \rightarrow +\infty} \int_{t_0}^t T_{A_0}(t-r) \lambda R_\lambda(A) \left[\int_{t_0}^r h(r, s) ds \right] dr = \frac{d}{dt} \int_{t_0}^t S_A(t-r) \left[\int_{t_0}^r h(r, s) ds \right] dr,$$

for all $t \geq t_0$. ■

Using Lemma 3.2 and Proposition 2.9 we can prove the following technical lemma.

Lemma 3.3 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Let $f \in C(\mathbb{R}, X)$. Define for each $\lambda > \omega$ and $(t, t_0) \in \Delta$*

$$v_\lambda(t, t_0) := \int_{t_0}^t U_B(t, s) \lambda R_\lambda(A) f(s) ds,$$

and

$$w(t, t_0) := \frac{d}{dt} \int_{t_0}^t S_A(t-s) f(s) ds = (S_A \diamond f(t_0 + \cdot))(t - t_0).$$

Then we have the following properties

i) For each $\lambda > \omega$ and $(t, t_0) \in \Delta$

$$v_\lambda(t, t_0) = \frac{d}{dt} \int_{t_0}^t S_A(t-r) B(r) v_\lambda(r, t_0) dr + \lambda R_\lambda(A) w(t, t_0), \quad \forall t \geq t_0.$$

ii) If in addition $\sup_{t \in \mathbb{R}} b(t) < +\infty$ then there exists a constant $\gamma > \max(0, \omega)$ such that for each $\lambda > \omega$ and $(t, t_0) \in \Delta$

$$\sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| \leq 2 \sup_{s \in [t_0, t]} e^{-\gamma s} \|\lambda R_\lambda(A) w(s, t_0)\|$$

and since $w(s, t_0) \in X_0$ we have

$$\|\lambda R_\lambda(A) w(s, t_0)\| \leq \frac{M|\lambda|}{\lambda - \omega} \|w(s, t_0)\|, \quad \forall s \in [t_0, t].$$

Proof. *Proof of i)* : By using formula (1.7) we obtain for each $\lambda > \omega$ and $t \geq t_0$

$$\begin{aligned} v_\lambda(t, t_0) &= \int_{t_0}^t T_{A_0}(t-s) \lambda R_\lambda(A) f(s) ds \\ &\quad + \int_{t_0}^t \left[\frac{d}{dt} \int_s^t S_A(t-r) B(r) U_B(r, s) \lambda R_\lambda(A) f(s) dr \right] ds. \end{aligned}$$

Now note that from Theorem 2.6 we have for each $\lambda > \omega$

$$\int_{t_0}^t T_{A_0}(t-s) \lambda R_\lambda(A) f(s) ds = \lambda R_\lambda(A) \frac{d}{dt} \int_{t_0}^t S_A(t-s) f(s) ds, \quad \forall t \geq t_0, \quad (3.2)$$

and from Lemma 3.2 with $h(r, s) = B(r)U_B(r, s)\lambda R_\lambda(A)f(s)$

$$\int_{t_0}^t \left[\frac{d}{dt} \int_s^t S_A(t-r)h(r, s)dr \right] ds = \frac{d}{dt} \int_{t_0}^t S_A(t-r)v_\lambda(r, t_0)dr, \quad \forall t \geq t_0. \quad (3.3)$$

Then *i)* follows by combining (3.2) and (3.3).

Proof of ii) : To do this we will make use of Proposition 2.9. Let $\varepsilon > 0$ be given such that

$$2\varepsilon \sup_{s \in \mathbb{R}} b(s) < \frac{1}{4}. \quad (3.4)$$

Let $\tau_\varepsilon > 0$ be given with $M\delta(\tau_\varepsilon) \leq \varepsilon$. By combining Proposition 2.9 together with *i)* we obtain for each $\lambda > \omega$ and $t \geq t_0$ that

$$\|v_\lambda(t, t_0)\| \leq C(\varepsilon, \gamma) \sup_{s \in [t_0, t]} \left[e^{\gamma(t-s)} b(s) \|v_\lambda(s, t_0)\| \right] + \|\lambda R_\lambda(A)w(t, t_0)\|,$$

whenever $\gamma \in (\omega, +\infty)$ with

$$C(\varepsilon, \gamma) := \frac{2\varepsilon \max(1, e^{-\gamma\tau_\varepsilon})}{1 - e^{(\omega-\gamma)\tau_\varepsilon}}, \quad (3.5)$$

so that

$$\sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| \leq C(\varepsilon, \gamma) \sup_{s \in \mathbb{R}} b(s) \sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| + \sup_{s \in [t_0, t]} \|\lambda R_\lambda(A)w(s, t_0)\|.$$

By using (3.5) and (3.4) it is easily seen that one can chose $\gamma > \max(0, \omega)$ large enough such that

$$0 \leq C(\varepsilon, \gamma) \sup_{s \in \mathbb{R}} b(s) < \frac{1}{2},$$

and *ii)* follows. ■

The following Lemma will be needed in sequel.

Lemma 3.4 *Let Assumptions 1.1 and 1.2 be satisfied. Then for each $a, c \in \mathbb{R}$ with $a < c$ and each $x \in X$, the map $t \rightarrow (S_A * x \mathbb{1}_{[a, c]}(\cdot))(t)$ is differentiable on $[0, +\infty)$ and*

$$\frac{d}{dt}(S_A * x \mathbb{1}_{[a, c]}(\cdot))(t) = \begin{cases} 0 & \text{if } c \leq 0 \text{ or } t \leq a, \\ S_A(t - a^+)x & \text{if } c > 0 \text{ and } t \in [a, c), \\ T_{A_0}(t - c)S_A(c - a^+)x & \text{if } c > 0 \text{ and } t \geq c, \end{cases}$$

with $a^+ := \max(0, a)$.

Proof. The proof is straightforward. ■

Now we have all the material in order to prove Theorem 1.6.

Proof of Theorem 1.6. Since the proof is trivial when $f(t) = 0$ it is sufficient to prove our theorem for $x_0 = 0$. Let $t_0 \in \mathbb{R}$ be fixed. Recalling for each $\lambda > \omega$

$$v_\lambda(t, t_0) = \int_{t_0}^t U_B(t, s) \lambda R_\lambda(A) f(s) ds, \quad \forall t \geq t_0,$$

we will show that the limit

$$\bar{v}(t, t_0) := \lim_{\lambda \rightarrow +\infty} v_\lambda(t, t_0), \quad \forall t \geq t_0, \quad (3.6)$$

is well defined and is an integrated solution of

$$\frac{dv(t)}{dt} = [A + B(t)]v(t) + f(t), \quad t \geq t_0 \text{ and } v(t_0) = 0. \quad (3.7)$$

First of all note that by Lemma 2.8, problem (3.7) admits a unique integrated solution $v(\cdot, t_0) \in C([t_0, +\infty), X_0)$ satisfying

$$v(t, t_0) = (S_A \diamond (Bv(\cdot, t_0))(t_0 + \cdot))(t - t_0) + (S_A \diamond f(t_0 + \cdot))(t - t_0), \quad \forall t \geq t_0, \quad (3.8)$$

where we have used the notation $(Bv(\cdot, t_0))(t) = B(t)v(t, t_0)$ for every $t \geq t_0$. Furthermore by Lemma 3.3 we also have for each $\lambda > \omega$ and each $t \geq t_0$

$$v_\lambda(t, t_0) = \frac{d}{dt} \int_{t_0}^t S_A(t - r) B(r) v_\lambda(r, t_0) dr + \lambda R_\lambda(A) w(t, t_0), \quad \forall t \geq t_0, \quad (3.9)$$

with

$$w(t, t_0) = \frac{d}{dt} \int_{t_0}^t S_A(t - s) f(s) ds = (S_A \diamond f(t_0 + \cdot))(t - t_0), \quad \forall t \geq t_0. \quad (3.10)$$

Then (3.8) and (3.9) rewrites, for each $\lambda > \omega$, as the following system

$$\begin{cases} v_\lambda(t, t_0) &= (S_A \diamond (Bv_\lambda(\cdot, t_0))(t_0 + \cdot))(t - t_0) + \lambda R_\lambda(A) w(t, t_0), \quad t \geq t_0 \\ v(t, t_0) &= (S_A \diamond (Bv(\cdot, t_0))(t_0 + \cdot))(t - t_0) + w(t, t_0), \quad t \geq t_0 \end{cases} \quad (3.11)$$

where we have used the notation $(Bv_\lambda(\cdot, t_0))(t) = B(t)v_\lambda(t, t_0)$ for every $t \geq t_0$.

Let $I \subset \mathbb{R}$ be a compact subset of \mathbb{R} . To show that (3.6) exists uniformly for $t \geq t_0$ in I , we will make use of Proposition 2.9.

We have from (3.11) that for every $\lambda > \omega$

$$v_\lambda(t, t_0) - v(t, t_0) = (S_A \diamond (B(v_\lambda(\cdot, t_0) - v(\cdot, t_0)))(t_0 + \cdot))(t - t_0) + [\lambda R_\lambda(A) - I]w(t), \quad \forall t \geq t_0 \quad (3.12)$$

with the notation

$$(B(v_\lambda(\cdot, t_0) - v(\cdot, t_0)))(t) := B(t)(v_\lambda(t, t_0) - v(t, t_0)), \quad \forall t \geq t_0.$$

Let $\varepsilon > 0$ be given such that

$$2\varepsilon \sup_{s \in I} b(s) < \frac{1}{4}. \quad (3.13)$$

Let $\tau_\varepsilon > 0$ be given with $M\delta(\tau_\varepsilon) \leq \varepsilon$. Then by using (3.12) and Proposition 2.9 we obtain for each $\lambda > \omega$ and each $t \geq t_0$ with $t, t_0 \in I$

$$\|v_\lambda(t, t_0) - v(t, t_0)\| \leq C(\varepsilon, \gamma) \sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} \left[e^{\gamma(t-s)} b(s) \|v_\lambda(s, t_0) - v(s, t_0)\| \right] + \|[\lambda R_\lambda(A) - I]w(t, t_0)\|,$$

whenever $\gamma \in (\omega, +\infty)$ with

$$C(\varepsilon, \gamma) := \frac{2\varepsilon \max(1, e^{-\gamma\tau_\varepsilon})}{1 - e^{(\omega-\gamma)\tau_\varepsilon}},$$

so that

$$\begin{aligned} \sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} e^{-\gamma s} \|v_\lambda(s, t_0) - v(s, t_0)\| &\leq C(\varepsilon, \gamma) \sup_{s \in I} b(s) \sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} e^{-\gamma s} \|v_\lambda(s, t_0) - v(s, t_0)\| \\ &\quad + \sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} \|[\lambda R_\lambda(A) - I]w(s, t_0)\|. \end{aligned}$$

By using (3.13) one can chose $\gamma > 0$ large enough such that

$$0 \leq C(\varepsilon, \gamma) \sup_{s \in I} b(s) < \frac{1}{2},$$

providing for all $\lambda > \omega$ that

$$\sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} e^{-\gamma s} \|v_\lambda(s, t_0) - v(s, t_0)\| \leq 2 \sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} e^{-\gamma s} \|[\lambda R_\lambda(A) - I]w(s, t_0)\|.$$

Hence recalling that the limit $\lim_{\lambda \rightarrow +\infty} \lambda R_\lambda(A)y = y$ is uniform on relatively compact sets of X_0 and by observing that $w(\cdot, \cdot)$ maps $I \times I$ into a relatively compact set of X_0 we obtain

$$\lim_{\lambda \rightarrow +\infty} \sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} e^{-\gamma s} \|[\lambda R_\lambda(A) - I]w(s, t_0)\| = 0$$

that is

$$\lim_{\lambda \rightarrow +\infty} \sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} e^{-\gamma s} \|v_\lambda(s, t_0) - v(s, t_0)\| = 0$$

and since I is bounded this implies

$$\lim_{\lambda \rightarrow +\infty} \sup_{\substack{s \geq t_0 \\ s, t_0 \in I}} \|v_\lambda(s, t_0) - v(s, t_0)\| = 0.$$

The proof is complete. ■

4 A uniform convergence results

Let $BUC(\mathbb{R}, X)$ be the space of bounded and uniformly continuous functions on \mathbb{R} . The next proposition gives a uniform approximation result subject to f belongs to an appropriate subspace of $BUC(\mathbb{R}, X)$.

Proposition 4.1 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Let $f \in BUC(\mathbb{R}, X)$ with relatively compact range. Then, for any fixed $t_0 > 0$ the limit

$$\lim_{\lambda \rightarrow +\infty} \int_{t-t_0}^t U_B(t, s) \lambda R_\lambda(A) f(s) ds,$$

exists uniformly for $t \in \mathbb{R}$.

Proof. Let $t_0 > 0$ be fixed. Recall that for each $\lambda > \omega$ we have

$$v_\lambda(t, t - t_0) = \int_{t-t_0}^t U_B(t, s) \lambda R_\lambda(A) f(s) ds, \quad \forall t \in \mathbb{R}.$$

Thus by using similar arguments in the proof of *ii*) in Lemma 3.3 we have for each $t \in \mathbb{R}$, each $\lambda > \omega$ and $\mu > \omega$

$$\sup_{s \in [t-t_0, t]} e^{-\gamma s} \|v_\lambda(s, s-t_0) - v_\mu(s, s-t_0)\| \leq 2 \sup_{s \in [t-t_0, t]} e^{-\gamma s} \|[\lambda R_\lambda(A) - \mu R_\mu(A)]w(s, s-t_0)\|,$$

with $\gamma > \max(0, \omega)$ (large enough) and

$$w(t_1, t_2) = (S_A \diamond f(t_2 + \cdot))(t_1 - t_2), \quad \forall (t_1, t_2) \in \Delta.$$

Hence for each $t \in \mathbb{R}$, each $\lambda > \omega$ and $\mu > \omega$

$$\begin{aligned} \|v_\lambda(t, t - t_0) - v_\mu(t, t - t_0)\| &\leq 2 \sup_{s \in [t-t_0, t]} e^{\gamma(t-s)} \|[\lambda R_\lambda(A) - \mu R_\mu(A)]w(s, s-t_0)\| \\ &\leq 2 e^{\gamma t_0} \sup_{s \in [t-t_0, t]} \|[\lambda R_\lambda(A) - \mu R_\mu(A)]w(s, s-t_0)\| \end{aligned}$$

Then to prove our proposition, it is sufficient to show that

$$\lim_{\lambda, \mu \rightarrow +\infty} \sup_{s \in \mathbb{R}} \|[\lambda R_\lambda(A) - \mu R_\mu(A)]w(s, s-t_0)\| = 0.$$

This can be achieved by proving that $w(\cdot, \cdot - t_0)$ maps \mathbb{R} in a relatively compact subset of X_0 . To do so we will prove that for any $\varepsilon > 0$, there exists a relatively compact set K such that

$$w(t, t - t_0) \in K + \varepsilon B_{X_0}(0, 1), \quad \forall t \in \mathbb{R},$$

for some constant $c > 0$ and $B_{X_0}(0, 1)$ the closed unit ball of X_0 .

Let $\varepsilon > 0$ be given and fixed. Then since f has its range in a relatively compact subset

of X , there exists $\eta = \frac{t_0}{n} > 0$, with $n \in \mathbb{N} \setminus \{0\}$ and a function $g : \mathbb{R} \rightarrow X$ such that g is constant on each interval $[k\eta, (k+1)\eta)$, $k \in \mathbb{Z}$. Moreover the range of g is contained in a finite set $K_0 \subset X$ and

$$\sup_{t \in \mathbb{R}} \|f(t) - g(t)\| \leq \varepsilon.$$

Note that g can be written as

$$g(t) = \sum_{k \in \mathbb{Z}} x_k \mathbb{1}_{[k\eta, k\eta + \eta)}(t), \quad \forall t \in \mathbb{R},$$

with $x_k \in K_0$ for all $k \in \mathbb{Z}$. Then by Lemma 3.4 it is easy to see that $t \rightarrow (S_A * g)(t)$ is differentiable on $[0, +\infty)$ and we can write

$$w(t, t - t_0) = (S_A \diamond g(t - t_0 + \cdot))(t_0) + (S_A \diamond (f - g)(t - t_0 + \cdot))(t_0), \quad \forall t \in \mathbb{R}.$$

Let $t \in \mathbb{R}$ be fixed. Note that one can write

$$t = k_0\eta + r, \quad \text{with } r \in [0, \eta) \text{ and } k_0 \in \mathbb{Z},$$

providing that (recalling $t_0 = n\eta$)

$$\begin{aligned} (S_A \diamond g(t - t_0 + \cdot))(t_0) &= \frac{d}{dt} \int_0^{t_0} S_A(t_0 - s)g(t - t_0 + s)ds \\ &= \frac{d}{dt} \int_{t-t_0}^t S_A(t - s)g(s)ds \\ &= \frac{d}{dt} \int_{(k_0-n)\eta+r}^{k_0\eta+r} S_A(t - s)g(s)ds \\ &= \sum_{i=0}^{n-1} \frac{d}{dt} \int_{(k_0-i-1)\eta+r}^{(k_0-i)\eta+r} S_A(k_0\eta + r - s)g(s)ds \\ &= \sum_{i=0}^{n-1} \frac{d}{dt} \left[\int_{(k_0-i-1)\eta+r}^{(k_0-i)\eta} S_A(k_0\eta + r - s)x_{k_0-i-1}ds \right. \\ &\quad \left. + \int_{(k_0-i)\eta}^{(k_0-i)\eta+r} S_A(k_0\eta + r - s)x_{k_0-i}ds \right] \end{aligned}$$

therefore we obtain

$$\begin{aligned}
(S_A \diamond g(t - t_0 + \cdot))(t_0) &= \sum_{i=0}^{n-1} [S_A(i\eta + \eta) - S_A(i\eta + r)] x_{k_0-i-1} \\
&\quad + \sum_{i=0}^{n-1} [S_A(i\eta + r) - S_A(i\eta)] x_{k_0-i} \\
&= \sum_{i=1}^n [S_A(i\eta) - S_A(i\eta - \eta + r)] x_{k_0-i} \\
&\quad + \sum_{i=0}^{n-1} [S_A(i\eta + r) - S_A(i\eta)] x_{k_0-i} \\
&= [S_A(n\eta) - S_A(n\eta - \eta + r)] x_{k_0-n} \\
&\quad + \sum_{i=1}^{n-1} [S_A(i\eta + r) - S_A(i\eta - \eta + r)] x_{k_0-i} + S_A(r) x_{k_0} \\
&= T_{A_0}(n\eta - \eta + r) S_A(\eta - r) x_{k_0-n} \\
&\quad + \sum_{i=1}^{n-1} T_{A_0}(i\eta - \eta + r) S_A(\eta) x_{k_0-i} + S_A(r) x_{k_0},
\end{aligned}$$

so that we can claim that $t \rightarrow (S_A \diamond g(t - t_0 + \cdot))(t_0)$ has its range in

$$K = \left\{ \sum_{k=0}^n T_{A_0}(s_k) S_A(l_k) x_k : 0 \leq s_k, l_k \leq t_0 \text{ and } x_k \in K_0, k = 0, \dots, n \right\}.$$

Then recalling that

$$(t, x) \in [0, +\infty) \times X \rightarrow S(t)x \text{ and } (t, x) \in [0, +\infty) \times X_0 \rightarrow T(t)x$$

are continuous, K is clearly compact.

To complete the proof it remains to give an estimate of $z(\cdot, \cdot - t_0)$ with

$$z(t_1, t_2) := (S_A \diamond (f - g)(t_2 + \cdot))(t_1 - t_2), \quad \forall (t_1, t_2) \in \Delta.$$

By using Proposition 2.9 one obtains

$$\|z(t_1, t_2)\| \leq C(1, \gamma_0) \sup_{t \in [0, t_1 - t_2]} e^{\gamma_0(t_1 - t_2 - t)} \|f(t_2 + t) - g(t_2 + t)\|, \quad \forall (t_1, t_2) \in \Delta.$$

with $\gamma_0 > \max(0, \omega)$, $M\delta(\tau_1) \leq 1$ and

$$C(1, \gamma_0) := \frac{2 \max(1, e^{-\gamma_0 \tau_1})}{1 - e^{(\omega - \gamma_0) \tau_1}}.$$

Therefore

$$\begin{aligned}
\sup_{(t_1, t_2) \in \Delta} \|z(t_1, t_2)\| &\leq C(1, \gamma_0) e^{\gamma_0(t_1 - t_2)} \sup_{t \in \mathbb{R}} \|f(t) - g(t)\| \\
&\leq C(1, \gamma_0) e^{\gamma_0(t_1 - t_2)} \varepsilon,
\end{aligned}$$

that is

$$\sup_{t \in \mathbb{R}} \|z(t, t - t_0)\| \leq C(1, \gamma_0) e^{\gamma_0 t_0} \varepsilon,$$

and the result follows. ■

5 Exponential dichotomy

In this section we consider the complete orbit of the Cauchy problem (1.1). Namely we consider a continuous map $u : (-\infty, +\infty) \rightarrow X_0$ as a mild solution of

$$\frac{du(t)}{dt} = (A + B(t))u(t) + f(t), \text{ for } t \in \mathbb{R}. \quad (5.1)$$

This part is devoted to the proof of Theorems 1.10 and 1.11. We will give necessary and sufficient condition for the evolution family $\{U_B(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(X_0)$ to have an exponential dichotomy. More precisely we will prove that the existence of exponential dichotomy for $\{U_B(t, s)\}_{(t,s) \in \Delta}$ is equivalent to the existence of integrated solution $u \in C(\mathbb{R}, X_0)$ for all f in an appropriate subspace of $C(\mathbb{R}, X)$.

In what follows when $\{U(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(Z)$ has an exponential dichotomy we define its associate Green's operator function by

$$\Gamma(t, s) := \begin{cases} U^+(t, s), & \text{if } t \geq s, \\ -U^-(s, t), & \text{if } t < s. \end{cases}$$

Remark 5.1 *It is well known that when $\{U(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(Z)$ has an exponential dichotomy then for each $x \in Z$, the map $(t, s) \in \mathbb{R}^2 \rightarrow U^-(t, s)x$ is continuous from \mathbb{R}^2 into Z (see [35, Lemma VI.9.15] or [18, Lemma 9.17]).*

Remark 5.2 *It is easy to obtain from condition i) in Definition 1.8*

$$\Pi^+(t)\Pi^-(t) = \Pi^-(t)\Pi^+(t) = 0_{\mathcal{L}(Z)}.$$

We also trivially have

$$U^+(t, t) = \Pi^+(t) \quad \text{and} \quad U^+(t, r)U^+(r, l) = U^+(t, l), \quad \forall t \geq r \geq l,$$

while

$$U^-(t, t) = \Pi^-(t) \quad \text{and} \quad U^-(t, r)U^-(r, l) = U^-(t, l), \quad \forall t, r, l \in \mathbb{R},$$

It follows that U^+ (respectively U^-) is a strongly continuous semiflow (respectively flow). One may also observe that

$$U^-(t, r)U(r, l) = U^-(t, l), \quad \forall (r, t), (r, l) \in \Delta$$

and

$$U^+(t, r)U(r, l) = U^+(t, l), \quad \forall (t, r), (r, l) \in \Delta.$$

Notation 5.3 *Let $(Z, \|\cdot\|)$ be a Banach space. The following weighted Banach spaces will be used in the sequel*

$$BC^\eta(\mathbb{R}, Z) := \left\{ f \in C(\mathbb{R}, Z) : \|f\|_\eta := \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|f(t)\|_Z < +\infty \right\}, \quad \eta \geq 0.$$

Note that we have the following continuous embedding

$$BC^{\eta_1}(\mathbb{R}, Z) \subseteq BC^{\eta_2}(\mathbb{R}, Z) \quad \text{if} \quad \eta_1 \leq \eta_2.$$

If $\eta = 0$ we set

$$BC(\mathbb{R}, Z) := \left\{ f \in C(\mathbb{R}, Z) : \|f\|_\infty := \sup_{t \in \mathbb{R}} \|f(t)\|_Z < +\infty \right\}, \quad \eta \geq 0.$$

and we define

$$C_0(\mathbb{R}, Z) := \left\{ f \in BC(\mathbb{R}, Z) : \lim_{t \rightarrow \pm\infty} f(t) = 0 \right\}.$$

The following result is well known in the context of exponential dichotomy. We refer for instance to [6, 25, 26].

Theorem 5.4 *Let Z be a Banach space. Let $\{U(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(Z)$ be an exponentially bounded evolution family. Consider the following integral equation*

$$u(t) = U(t, t_0)u(t_0) + \int_{t_0}^t U(t, s)f(s)ds, \quad (t, t_0) \in \Delta. \quad (5.2)$$

Then the following properties are equivalent

- i) $\{U(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(Z)$ has an exponential dichotomy.
- ii) Let $\mathcal{F}(\mathbb{R}, Z)$ be the space $BC(\mathbb{R}, Z)$ or $C_0(\mathbb{R}, Z)$. Then for any $f \in \mathcal{F}(\mathbb{R}, Z)$ there exists a unique solution $u \in \mathcal{F}(\mathbb{R}, Z)$ of (5.2).

Moreover if $\{U(t, s)\}_{(t,s) \in \Delta}$ has an exponential dichotomy then for each $f \in \mathcal{F}(\mathbb{R}, Z)$ the unique solution $u \in \mathcal{F}(\mathbb{R}, Z)$ of (5.2) is given by

$$u(t) = \int_{-\infty}^{+\infty} \Gamma(t, s)f(s)ds, \quad \forall t \in \mathbb{R},$$

where $\{\Gamma(t, s)\}_{(t,s) \in \mathbb{R}^2} \subset \mathcal{L}(Z)$ is the Green's operator function associated to $\{U(t, s)\}_{(t,s) \in \Delta}$.

In what follow we will give an analogue of Theorem 5.4 for the evolution family $\{U_B(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(X_0)$. To do so we will first prove some estimates.

Proposition 5.5 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Then there exists a non decreasing function $\delta^* : [0, +\infty) \rightarrow [0, +\infty)$ with $\delta^*(t) \rightarrow 0$ as $t \rightarrow 0^+$ such that for each $f \in C(\mathbb{R}, X)$ and $\lambda > w + 1$ the map

$$v_\lambda(t, t_0) = \int_{t_0}^t U_B(t, s)\lambda R_\lambda(A)f(s)ds, \quad (t, t_0) \in \Delta,$$

satisfies

$$\|v_\lambda(t, t_0)\| \leq \delta^*(t - t_0) \sup_{s \in [t_0, t]} \|f(s)\|, \quad \forall (t, t_0) \in \Delta.$$

Proof. Let $\lambda > \omega$ be given. Thus by Lemma 3.3 there exists $\gamma > \max(0, \omega)$ large enough (independent of t_0) such that for each $t \geq t_0$

$$\sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| \leq 2 \sup_{s \in [t_0, t]} e^{-\gamma s} \|\lambda R_\lambda(A)w(s, t_0)\|,$$

with

$$w(t_1, t_2) = (S_A \diamond f(t_2 + \cdot))(t_1 - t_2), \quad \forall (t_1, t_2) \in \Delta.$$

Since $w(t_1, t_2) \in X_0$ for all $(t_1, t_2) \in \Delta$ and by Assumption 1.3

$$\|w(t_1, t_2)\| \leq \delta(t_2 - t_1) \sup_{s \in [t_1, t_2]} \|f(s)\|, \quad \forall (t_1, t_2) \in \Delta,$$

it follows that for each $\lambda > \omega$ and $t \geq t_0$

$$\begin{aligned} \sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| &\leq 2 \sup_{s \in [t_0, t]} e^{-\gamma s} \|\lambda R_\lambda(A_0)w(s, t_0)\| \\ &\leq 2 \frac{M|\lambda|}{\lambda - \omega} \sup_{s \in [t_0, t]} e^{-\gamma s} \|w(s, t_0)\| \\ &\leq 2 \frac{M|\lambda|}{\lambda - \omega} \sup_{s \in [t_0, t]} \left[e^{-\gamma s} \delta(s - t_0) \sup_{l \in [t_0, s]} \|f(l)\| \right]. \end{aligned}$$

Then by using the fact that δ is non decreasing and $\gamma > 0$ we obtain for each $\lambda > \omega$ and $t \geq t_0$

$$\sup_{s \in [t_0, t]} e^{-\gamma s} \|v_\lambda(s, t_0)\| \leq 2 \frac{M|\lambda|}{\lambda - \omega} e^{-\gamma t_0} \delta(t - t_0) \sup_{s \in [t_0, t]} \|f(s)\|, \quad \forall t \geq t_0,$$

providing that

$$\|v_\lambda(t, t_0)\| \leq 2 \frac{M|\lambda|}{\lambda - \omega} e^{\gamma(t-t_0)} \delta(t - t_0) \sup_{s \in [t_0, t]} \|f(s)\|, \quad \forall t \geq t_0.$$

The proof is easily completed by using the fact that

$$\lambda > \omega + 1 \Rightarrow \frac{|\lambda|}{\lambda - \omega} < 1 + |\omega|.$$

■

In the rest of this paper, the following assumption will be used.

Assumption 5.6 Assume that $\{U_B(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(X_0)$ has an exponential dichotomy with exponent $\beta > 0$, constant $\kappa \geq 1$ and strongly continuous projectors $\{\Pi_B^+(t)\}_{t \in \mathbb{R}} \subset \mathcal{L}(X_0)$ and $\{\Pi_B^-(t)\}_{t \in \mathbb{R}} \subset \mathcal{L}(X_0)$.

Note that if $\{U_B(t, s)\}_{(t,s) \in \Delta}$ has an exponential dichotomy then combining Remark 5.2 and condition *iv*) in Definition 1.8 we have

$$\sup_{t \in \mathbb{R}} \|\Pi_B^+(t)\|_{\mathcal{L}(Z)} \leq \kappa \text{ and } \sup_{t \in \mathbb{R}} \|\Pi_B^-(t)\|_{\mathcal{L}(Z)} \leq \kappa. \quad (5.3)$$

Proposition 5.7 *Let Assumption 1.1 be satisfied. Let $\{U(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(X_0)$ be a given evolution family such that there exist $\widehat{M} \geq 1$, $\widehat{\omega} \in \mathbb{R}$ and*

$$\|U(t, s)\|_{\mathcal{L}(X_0)} \leq \widehat{M}e^{\widehat{\omega}(t-s)}, \quad \forall (t, s) \in \Delta.$$

Assume that for each $f \in C(\mathbb{R}, X)$ the map

$$v_\lambda(t, t_0) = \int_{t_0}^t U(t, s) \lambda R_\lambda(A) f(s) ds, \quad (t, t_0) \in \Delta,$$

satisfies

$$\|v_\lambda(t, t_0)\| \leq \delta^{**}(t - t_0) \sup_{s \in [t_0, t]} \|f(s)\|, \quad \forall (t, t_0) \in \Delta,$$

*with $\delta^{**} : [0, +\infty) \rightarrow [0, +\infty)$ a non decreasing function such that $\delta^{**}(t) \rightarrow 0$ as $t \rightarrow 0^+$. Let $\varepsilon > 0$ be given and fixed. Then, for each $\tau_\varepsilon > 0$ satisfying $\widehat{M}\delta^{**}(\tau_\varepsilon) \leq \varepsilon$ and each $\lambda > \omega + 1$ we have*

$$\|v_\lambda(t, t_0)\| \leq \widetilde{C}(\varepsilon, \gamma, \widehat{\omega}, \widehat{M}) \sup_{s \in [t_0, t]} e^{\gamma(t-s)} \|f(s)\|, \quad \forall (t, t_0) \in \Delta,$$

whenever $\gamma > \widehat{\omega}$ and $f \in C(\mathbb{R}, X)$ with

$$\widetilde{C}(\varepsilon, \gamma, \widehat{\omega}, \widehat{M}) := \widehat{M}e^{\max(0, \widehat{\omega})\tau_\varepsilon} \frac{2\varepsilon \max(1, e^{-\gamma\tau_\varepsilon})}{1 - e^{(\widehat{\omega} - \gamma)\tau_\varepsilon}}.$$

Proof. Without loss of generality we can assume that $t_0 = 0$. Let $\tau_\varepsilon > 0$ be given such that

$$\widehat{M}\delta^{**}(s) \leq \varepsilon, \quad \forall s \in [0, \tau_\varepsilon].$$

Let $t \geq 0$ be fixed. Then since we can write $t = n\tau_\varepsilon + \theta$ with $\theta \in [0, \tau_\varepsilon)$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned} v_\lambda(t, 0) &= \int_0^t U(t, s) \lambda R_\lambda(A) f(s) ds \\ &= \sum_{k=0}^{n-1} \int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} U(t, s) \lambda R_\lambda(A) f(s) ds + \int_{n\tau_\varepsilon}^t U(t, s) \lambda R_\lambda(A) f(s) ds \\ &= \sum_{k=0}^{n-1} U(t, (k+1)\tau_\varepsilon) \int_{k\tau_\varepsilon}^{(k+1)\tau_\varepsilon} U((k+1)\tau_\varepsilon, s) \lambda R_\lambda(A) f(s) ds \\ &\quad + U(t, n\tau_\varepsilon) \int_{n\tau_\varepsilon}^t U(n\tau_\varepsilon, s) \lambda R_\lambda(A) f(s) ds \\ &= \sum_{k=0}^{n-1} U(t, (k+1)\tau_\varepsilon) v_\lambda((k+1)\tau_\varepsilon, k\tau_\varepsilon) + U(t, n\tau_\varepsilon) v_\lambda(t, n\tau_\varepsilon), \end{aligned}$$

so that

$$v_\lambda(t, 0) = U(t, n\tau_\varepsilon) \sum_{k=0}^{n-1} U(n\tau_\varepsilon, (k+1)\tau_\varepsilon) v_\lambda((k+1)\tau_\varepsilon, k\tau_\varepsilon) + U(t, n\tau_\varepsilon) v_\lambda(t, n\tau_\varepsilon). \quad (5.4)$$

Next observe that for all $(r_0, r_1) \in \Delta$ and $r \geq r_0$ with $0 \leq r_0 - r_1 \leq \tau_\varepsilon$ we have

$$\begin{aligned} \|U(r, r_0)v_\lambda(r_0, r_1)\| &\leq \widehat{M}e^{\widehat{\omega}(r-r_0)}\|v_\lambda(r_0, r_1)\| \\ &\leq e^{\widehat{\omega}(r-r_0)}\widehat{M}\delta^*(r_0 - r_1) \sup_{s \in [r_1, r_0]} \|f(s)\| \\ &\leq e^{\widehat{\omega}(r-r_0)}\varepsilon \sup_{s \in [r_1, r_0]} \|f(s)\|. \end{aligned} \quad (5.5)$$

Let $\gamma > \widehat{\omega}$ be fixed. Set $\varepsilon_1 := \max(1, e^{-\gamma\tau_\varepsilon})$. Let $k \in \mathbb{N}$ and $r \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$ be given and fixed.

Then if $\gamma \geq 0$ we have

$$\varepsilon \sup_{s \in [k\tau_\varepsilon, r]} \|f(s)\| = \varepsilon \sup_{s \in [k\tau_\varepsilon, r]} e^{-\gamma s} e^{\gamma s} \|f(s)\| \leq \varepsilon_1 e^{\gamma r} \sup_{s \in [k\tau_\varepsilon, r]} e^{-\gamma s} \|f(s)\|, \quad (5.6)$$

while if $\gamma < 0$

$$\begin{aligned} \varepsilon \sup_{s \in [k\tau_\varepsilon, r]} \|f(s)\| &= \varepsilon \sup_{s \in [k\tau_\varepsilon, r]} e^{-\gamma s} e^{\gamma s} \|f(s)\| \\ &\leq \varepsilon e^{\gamma k\tau_\varepsilon} \sup_{s \in [k\tau_\varepsilon, r]} e^{-\gamma s} \|f(s)\| \\ &\leq \varepsilon e^{\gamma r} e^{-\gamma(r-k\tau_\varepsilon)} \sup_{s \in [k\tau_\varepsilon, r]} e^{-\gamma s} \|f(s)\| \\ &\leq \varepsilon e^{\gamma r} e^{-\gamma\tau_\varepsilon} \sup_{s \in [k\tau_\varepsilon, r]} e^{-\gamma s} \|f(s)\| \\ &\leq \varepsilon_1 e^{\gamma r} \sup_{s \in [k\tau_\varepsilon, r]} e^{-\gamma s} \|f(s)\|. \end{aligned}$$

Therefore for each $k \in \mathbb{N}$, each $r \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$ and $\gamma > \widehat{\omega}$ we obtain

$$\varepsilon \sup_{s \in [k\tau_\varepsilon, r]} \|f(s)\| \leq \varepsilon_1 e^{\gamma r} \sup_{s \in [k\tau_\varepsilon, r]} e^{-\gamma s} \|f(s)\|. \quad (5.7)$$

By (5.5) and (5.7) we obtain for each $k \in \mathbb{N}$, each $r \geq (k+1)\tau_\varepsilon$ and $\gamma > \widehat{\omega}$

$$\|U(r, (k+1)\tau_\varepsilon)v_\lambda((k+1)\tau_\varepsilon, k\tau_\varepsilon)\| \leq e^{-(\beta+\gamma)(r-(k+1)\tau_\varepsilon)}\varepsilon_1 e^{\gamma r} \sup_{s \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]} e^{-\gamma s} \|f(s)\|.$$

$$\|U(r, (k+1)\tau_\varepsilon)v_\lambda((k+1)\tau_\varepsilon, k\tau_\varepsilon)\| \leq e^{(\widehat{\omega}-\gamma)(r-(k+1)\tau_\varepsilon)}\varepsilon_1 e^{\gamma r} \sup_{s \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]} e^{-\gamma s} \|f(s)\|. \quad (5.8)$$

Since $t - n\tau_\varepsilon \in [0, \tau_\varepsilon]$ we have from (5.5) and (5.7) that

$$\begin{aligned} \|U(t, n\tau_\varepsilon)v_\lambda(t, n\tau_\varepsilon)\| &\leq e^{\widehat{\omega}(t-n\tau_\varepsilon)}\varepsilon \sup_{s \in [n\tau_\varepsilon, t]} \|f(s)\| \\ &\leq e^{\widehat{\omega}(t-n\tau_\varepsilon)}\varepsilon_1 e^{\gamma t} \sup_{s \in [n\tau_\varepsilon, t]} e^{-\gamma s} \|f(s)\|, \\ &\leq e^{\max(0, \widehat{\omega})\tau_\varepsilon}\varepsilon_1 e^{\gamma t} \sup_{s \in [n\tau_\varepsilon, t]} e^{-\gamma s} \|f(s)\|, \end{aligned}$$

and by using (5.4) and (5.8) we obtain

$$\begin{aligned}
\|v_\lambda(t, 0)\| &\leq \widehat{M}e^{\widehat{\omega}(t-n\tau_\varepsilon)} \sum_{k=0}^{n-1} \|U(n\tau_\varepsilon, (k+1)\tau_\varepsilon)v_\lambda((k+1)\tau_\varepsilon, k\tau_\varepsilon)\| + \|U(t, n\tau_\varepsilon)v_\lambda(t, n\tau_\varepsilon)\| \\
&\leq \widehat{M}e^{\widehat{\omega}(t-n\tau_\varepsilon)} \sum_{k=0}^{n-1} e^{(\widehat{\omega}-\gamma)(n\tau_\varepsilon-(k+1)\tau_\varepsilon)} \varepsilon_1 e^{\gamma n\tau_\varepsilon} \sup_{s \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]} e^{-\gamma s} \|f(s)\| \\
&\quad + e^{\max(0, \widehat{\omega})\tau_\varepsilon} \varepsilon_1 e^{\gamma t} \sup_{s \in [n\tau_\varepsilon, t]} e^{-\gamma s} \|f(s)\| \\
&\leq \widehat{M}e^{\widehat{\omega}(t-n\tau_\varepsilon)} e^{\gamma n\tau_\varepsilon} \left[\sum_{k=0}^{n-1} e^{(\widehat{\omega}-\gamma)(n-1-k)\tau_\varepsilon} \right] \varepsilon_1 \sup_{s \in [0, t]} e^{-\gamma s} \|f(s)\| \\
&\quad + e^{\max(0, \widehat{\omega})\tau_\varepsilon} \varepsilon_1 e^{\gamma t} \sup_{s \in [n\tau_\varepsilon, t]} e^{-\gamma s} \|f(s)\| \\
&\leq \widehat{M}e^{(\widehat{\omega}-\gamma)(t-n\tau_\varepsilon)} e^{\gamma t} \left[\sum_{k=0}^{n-1} e^{(\widehat{\omega}-\gamma)k\tau_\varepsilon} \right] \varepsilon_1 \sup_{s \in [0, t]} e^{-\gamma s} \|f(s)\| \\
&\quad + e^{\max(0, \widehat{\omega})\tau_\varepsilon} \varepsilon_1 e^{\gamma t} \sup_{s \in [0, t]} e^{-\gamma s} \|f(s)\|
\end{aligned}$$

Then since $\widehat{\omega} - \gamma < 0$ we obtain

$$\begin{aligned}
\|v_\lambda(t, 0)\| &\leq \widehat{M}e^{\max(0, \widehat{\omega})\tau_\varepsilon} e^{\gamma t} \left[1 + \sum_{k=0}^{+\infty} (e^{(\widehat{\omega}-\gamma)\tau_\varepsilon})^k \right] \varepsilon_1 \sup_{s \in [0, t]} e^{-\gamma s} \|f(s)\| \\
&\leq \widehat{M}e^{\max(0, \widehat{\omega})\tau_\varepsilon} e^{\gamma t} \left[\frac{2}{1 - e^{(\widehat{\omega}-\gamma)\tau_\varepsilon}} \right] \varepsilon_1 \sup_{s \in [0, t]} e^{-\gamma s} \|f(s)\|.
\end{aligned}$$

The proof is complete. ■

As a direct consequence of Propositions 5.7 and 5.5 we obtain the following result.

Proposition 5.8 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Let $\varepsilon > 0$ be given and fixed. Then, for each $\tau_\varepsilon > 0$ satisfying $\kappa\delta^(\tau_\varepsilon) \leq \varepsilon$ and each $\lambda > \omega + 1$ the map*

$$v_\lambda(t, t_0) = \int_{t_0}^t U_B(t, s) \lambda R_\lambda(A) f(s) ds, \quad (t, t_0) \in \Delta,$$

satisfies

$$\|\Pi^+(t)v_\lambda(t, t_0)\| \leq \widehat{C}(\varepsilon, \gamma) \sup_{s \in [t_0, t]} e^{\gamma(t-s)} \|f(s)\|, \quad \forall (t, t_0) \in \Delta,$$

whenever $\gamma > -\beta$ and $f \in C(\mathbb{R}, X)$ with

$$\widehat{C}(\varepsilon, \gamma) := \kappa \frac{2\varepsilon \max(1, e^{-\gamma\tau_\varepsilon})}{1 - e^{-(\beta+\gamma)\tau_\varepsilon}}. \quad (5.9)$$

Proposition 5.9 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Let $\varepsilon > 0$ be given and fixed. Then, for each $\tau_\varepsilon > 0$ satisfying $\kappa\delta^(\tau_\varepsilon) \leq \varepsilon$ and each $\lambda > \omega + 1$ the map*

$$v_\lambda(t, t_0) = \int_{t_0}^t U_B(t, s) \lambda R_\lambda(A) f(s) ds, \quad (t, t_0) \in \Delta,$$

satisfies

$$\|U_B^-(t_0, t) v_\lambda(t, t_0)\| \leq \widehat{C}(\varepsilon, \gamma) \sup_{s \in [t_0, t]} e^{\gamma(s-t_0)} \|f(s)\|, \quad \forall (t, t_0) \in \Delta,$$

whenever $\gamma > -\beta$ and $f \in C(\mathbb{R}, X)$ with $\widehat{C}(\varepsilon, \gamma)$ defined in (5.9).

Proof. Let $(t, t_0) \in \Delta$ be given. Without loss of generality one can assume that $t = 0$. From now one fix $t_0 \leq 0$. Let $\tau_\varepsilon > 0$ be given such that

$$\kappa\delta^*(s) \leq \varepsilon, \quad \forall s \in [0, \tau_\varepsilon].$$

Then since we can write $t_0 = -n\tau_\varepsilon - \theta$ with $\theta \in [0, \tau_\varepsilon)$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned} U_B^-(0, t_0) v_\lambda(0, t_0) &= \int_{t_0}^0 U_B^-(0, s) \lambda R_\lambda(A) f(s) ds \\ &= \sum_{k=0}^{n-1} \int_{-(k+1)\tau_\varepsilon}^{-k\tau_\varepsilon} U_B^-(0, s) \lambda R_\lambda(A) f(s) ds \\ &\quad + \int_{t_0}^{-n\tau_\varepsilon} U_B^-(0, s) \lambda R_\lambda(A) f(s) ds \\ &= \sum_{k=0}^{n-1} U_B^-(0, -k\tau_\varepsilon) \int_{-(k+1)\tau_\varepsilon}^{-k\tau_\varepsilon} U_B^-(-k\tau_\varepsilon, s) \lambda R_\lambda(A) f(s) ds \\ &\quad + U_B^-(0, -n\tau_\varepsilon) \int_{t_0}^{-n\tau_\varepsilon} U_B^-(-n\tau_\varepsilon, s) \lambda R_\lambda(A) f(s) ds, \end{aligned}$$

so that

$$\Pi^-(0) v_\lambda(0, t_0) = \sum_{k=0}^{n-1} U_B^-(0, -k\tau_\varepsilon) v_\lambda(-k\tau_\varepsilon, -(k+1)\tau_\varepsilon) + U_B^-(0, -n\tau_\varepsilon) v_\lambda(-n\tau_\varepsilon, t_0). \quad (5.10)$$

Since $U_B^-(0, t_0)$ is invertible from $\Pi^-(t_0)(X_0)$ into $\Pi^-(0)(X_0)$ with inverse $U_B^-(t_0, 0)$, by applying $U_B^-(t_0, 0)$ to (5.10) we obtain

$$U_B^-(t_0, 0) v_\lambda(0, t_0) = \sum_{k=0}^{n-1} U_B^-(t_0, -k\tau_\varepsilon) v_\lambda(-k\tau_\varepsilon, -(k+1)\tau_\varepsilon) + U_B^-(t_0, -n\tau_\varepsilon) v_\lambda(-n\tau_\varepsilon, t_0).$$

and by using the evolution property of U_B^- it follows that

$$\begin{aligned} U_B^-(t_0, 0)v_\lambda(0, t_0) &= U_B^-(t_0, -n\tau_\varepsilon) \sum_{k=0}^{n-1} U_B^-(-n\tau_\varepsilon, -k\tau_\varepsilon)v_\lambda(-k\tau_\varepsilon, -(k+1)\tau_\varepsilon) \\ &\quad + U_B^-(t_0, -n\tau_\varepsilon)v_\lambda(-n\tau_\varepsilon, t_0). \end{aligned} \quad (5.11)$$

Next observe that for all $(r_0, r_1) \in \Delta$ and $r \leq r_1$ with $0 \leq r_0 - r_1 \leq \tau_\varepsilon$ we have

$$\begin{aligned} \|U_B^-(r, r_0)v_\lambda(r_0, r_1)\| &\leq \kappa e^{-\beta(r_0-r)} \|v_\lambda(r_0, r_1)\| \\ &\leq e^{-\beta(r_0-r)} \kappa \delta^*(r_0 - r_1) \sup_{s \in [r_1, r_0]} \|f(s)\| \\ &\leq e^{-\beta(r_0-r)} \varepsilon \sup_{s \in [r_1, r_0]} \|f(s)\|. \end{aligned} \quad (5.12)$$

Let $\gamma > -\beta$ be fixed. Set $\varepsilon_1 := \max(1, e^{-\gamma\tau_\varepsilon})$. Let $k \in \mathbb{N}$ and $r \in [-(k+1)\tau_\varepsilon, -k\tau_\varepsilon]$ be given and fixed.

Then if $\gamma \geq 0$ we have

$$\varepsilon \sup_{s \in [r, -k\tau_\varepsilon]} \|f(s)\| = \varepsilon \sup_{s \in [r, -k\tau_\varepsilon]} e^{-\gamma s} e^{\gamma s} \|f(s)\| \leq \varepsilon_1 e^{-\gamma r} \sup_{s \in [r, -k\tau_\varepsilon]} e^{\gamma s} \|f(s)\|,$$

while if $\gamma < 0$

$$\begin{aligned} \varepsilon \sup_{s \in [r, -k\tau_\varepsilon]} \|f(s)\| &= \varepsilon \sup_{s \in [r, -k\tau_\varepsilon]} e^{-\gamma s} e^{\gamma s} \|f(s)\| \\ &\leq \varepsilon e^{\gamma k\tau_\varepsilon} \sup_{s \in [r, -k\tau_\varepsilon]} e^{\gamma s} \|f(s)\| \\ &\leq \varepsilon e^{-\gamma r} e^{\gamma(r+k\tau_\varepsilon)} \sup_{s \in [r, -k\tau_\varepsilon]} e^{\gamma s} \|f(s)\| \\ &\leq \varepsilon e^{-\gamma r} e^{-\gamma\tau_\varepsilon} \sup_{s \in [r, -k\tau_\varepsilon]} e^{\gamma s} \|f(s)\| \\ &\leq \varepsilon_1 e^{-\gamma r} \sup_{s \in [r, -k\tau_\varepsilon]} e^{\gamma s} \|f(s)\|. \end{aligned}$$

Therefore for each $k \in \mathbb{N}$, each $r \in [-(k+1)\tau_\varepsilon, -k\tau_\varepsilon]$ and $\gamma > -\beta$ we obtain

$$\varepsilon \sup_{s \in [r, -k\tau_\varepsilon]} \|f(s)\| \leq \varepsilon_1 e^{-\gamma r} \sup_{s \in [r, -k\tau_\varepsilon]} e^{\gamma s} \|f(s)\|. \quad (5.13)$$

By (5.12) and (5.13) we obtain for each $k \in \mathbb{N}$, each $r \leq -(k+1)\tau_\varepsilon$ and $\gamma > -\beta$ we obtain

$$\|U_B^-(r, -k\tau_\varepsilon)v_\lambda(-k\tau_\varepsilon, -(k+1)\tau_\varepsilon)\| \leq e^{(\beta+\gamma)(r+(k+1)\tau_\varepsilon)} e^{-\gamma r} \varepsilon_1 \sup_{s \in [-(k+1)\tau_\varepsilon, -k\tau_\varepsilon]} e^{\gamma s} \|f(s)\|.$$

Since $-n\tau_\varepsilon - t_0 = \theta \in [0, \tau_\varepsilon]$ we obtain from (5.11) and (5.13)

$$\begin{aligned} \|U_B^-(t_0, -n\tau_\varepsilon)v_\lambda(-n\tau_\varepsilon, t_0)\| &\leq e^{\beta(t_0+n\tau_\varepsilon)} \varepsilon \sup_{s \in [t_0, -n\tau_\varepsilon]} \|f(s)\| \\ &\leq e^{\beta(t_0+n\tau_\varepsilon)} \varepsilon_1 e^{-\gamma t_0} \sup_{s \in [t_0, -n\tau_\varepsilon]} e^{\gamma s} \|f(s)\|, \end{aligned} \quad (5.14)$$

and by using (5.11) and (5.14) it follows that

$$\begin{aligned}
\|U_B^-(t_0, 0)\Pi^-(0)v_\lambda(0, t_0)\| &\leq \kappa e^{\beta(t_0+n\tau_\varepsilon)} \left[\sum_{k=0}^{n-1} e^{(\beta+\gamma)(-n+k+1)\tau_\varepsilon} \varepsilon_1 \sup_{s \in [-(k+1)\tau_\varepsilon, -k\tau_\varepsilon]} e^{\gamma s} \|f(s)\| \right. \\
&\quad \left. + e^{\beta(t_0+n\tau_\varepsilon)} \varepsilon_1 e^{-\gamma t_0} \sup_{s \in [t_0, -n\tau_\varepsilon]} e^{\gamma s} \|f(s)\| \right] \\
&\leq \kappa e^{\beta(t_0+n\tau_\varepsilon)} e^{\gamma n\tau_\varepsilon} \left[\sum_{k=0}^{n-1} e^{(\beta+\gamma)(-n+k+1)\tau_\varepsilon} \varepsilon_1 \sup_{s \in [t_0, 0]} e^{\gamma s} \|f(s)\| \right. \\
&\quad \left. + e^{\beta(t_0+n\tau_\varepsilon)} \varepsilon_1 e^{-\gamma t_0} \sup_{s \in [t_0, 0]} e^{\gamma s} \|f(s)\| \right] \\
&\leq \kappa e^{(\beta+\gamma)(t_0+n\tau_\varepsilon)} e^{-\gamma t_0} \left[\sum_{k=-n+1}^0 (e^{(\beta+\gamma)\tau_\varepsilon})^k \varepsilon_1 \sup_{s \in [t_0, 0]} e^{\gamma s} \|f(s)\| \right. \\
&\quad \left. + e^{\beta(t_0+n\tau_\varepsilon)} \varepsilon_1 e^{-\gamma t_0} \sup_{s \in [t_0, 0]} e^{\gamma s} \|f(s)\| \right].
\end{aligned}$$

Finally since $\gamma + \beta > 0$ and $t_0 + n\tau_\varepsilon < 0$ we get

$$\begin{aligned}
\|U_B^-(t_0, 0)\Pi^-(0)v_\lambda(0, t_0)\| &\leq \kappa \left[1 + \sum_{k=-n+1}^0 (e^{(\beta+\gamma)\tau_\varepsilon})^k \right] \varepsilon_1 e^{-\gamma t_0} \sup_{s \in [t_0, 0]} e^{\gamma s} \|f(s)\| \\
&\leq \kappa \left[1 + \sum_{k=-\infty}^0 (e^{(\beta+\gamma)\tau_\varepsilon})^k \right] \varepsilon_1 e^{-\gamma t_0} \sup_{s \in [t_0, 0]} e^{\gamma s} \|f(s)\| \\
&\leq \kappa \left[\frac{2}{1 - e^{-(\beta+\gamma)}} \right] \varepsilon_1 e^{-\gamma t_0} \sup_{s \in [t_0, 0]} e^{\gamma s} \|f(s)\|.
\end{aligned}$$

■

Lemma 5.10 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Let $\eta \in [0, \beta)$ be given. Then for each $\lambda > \omega + 1$, each $f \in BC^\eta(\mathbb{R}, X)$ and $t \in \mathbb{R}$

$$\mathcal{J}_\lambda^+(f)(t) := \lim_{t_0 \rightarrow -\infty} \int_{t_0}^t U_B^+(t, s) \lambda R_\lambda(A) f(s) ds := \int_{-\infty}^t U_B^+(t, s) \lambda R_\lambda(A) f(s) ds, \quad (5.15)$$

exists. Moreover the following properties hold

- i) *For each $\eta \in [0, \beta)$ and each $\lambda > \omega + 1$, \mathcal{J}_λ^+ is a bounded linear operator from $BC^\eta(\mathbb{R}, X)$ into itself. More precisely for any $\nu \in (-\beta, 0)$*

$$\|\mathcal{J}_\lambda^+(f)\|_\eta \leq \widehat{C}(1, \nu) \|f\|_\eta, \quad \forall f \in BC^\eta(\mathbb{R}, X) \quad \text{with } \eta \in [0, -\nu],$$

where $\widehat{C}(1, \nu)$ is the constant introduced in Proposition 5.8.

ii) For each $\eta \in [0, \beta)$, each $\lambda > \omega + 1$ and each $f \in BC^\eta(\mathbb{R}, X)$ we have

$$\mathcal{J}_\lambda^+(f)(t) = U_B^+(t, l)\mathcal{J}_\lambda^+(f)(l) + \int_l^t U_B^+(t, s)\lambda R_\lambda(A)f(s)ds, \quad \forall (t, l) \in \Delta. \quad (5.16)$$

Proof. Let $\eta \in [0, \beta)$ be given. Let $\lambda > \omega + 1$ be given and fixed. Recall

$$v_\lambda(t, t_0) := \int_{t_0}^t U_B(t, s)\lambda R_\lambda(A)f(s)ds, \quad \forall (t, t_0) \in \Delta,$$

and observe that

$$\Pi^+(t)v_\lambda(t, t_0) = \int_{t_0}^t U_B^+(t, s)\lambda R_\lambda(A)f(s)ds, \quad \forall (t, t_0) \in \Delta,$$

and

$$\mathcal{J}_\lambda^+(f)(t) = \lim_{t_0 \rightarrow -\infty} \Pi^+(t)v_\lambda(t, t_0), \quad \forall t \in \mathbb{R}.$$

To prove the existence of the limit, we will show that for each fixed $t \in \mathbb{R}$, $\{\Pi^+(t)v_\lambda(t, t_0)\}_{t_0 \leq t}$ is a Cauchy sequence. Fix $t \in \mathbb{R}$. Let $f \in BC^\eta(\mathbb{R}, X)$ be given. Let $t_0, r \in \mathbb{R}$ such that $t_0 \leq r \leq t$. Then we have

$$\begin{aligned} \Pi^+(t)v_\lambda(t, t_0) &= U_B^+(t, r) \int_{t_0}^r U_B^+(r, s)\lambda R_\lambda(A)f(s)ds + \int_r^t U_B^+(t, s)\lambda R_\lambda(A)f(s)ds \\ &= U_B^+(t, r)v_\lambda(r, t_0) + \Pi^+(t)v_\lambda(t, r) \end{aligned}$$

and

$$\Pi^+(t)v_\lambda(t, t_0) - \Pi^+(t)v_\lambda(t, r) = U_B(t, r)\Pi^+(r)v_\lambda(r, t_0). \quad (5.17)$$

Hence by Proposition 5.8 we can find a constant $\widehat{C}(1, \gamma) > 0$ with $\gamma \in (-\beta, -\eta)$ such that

$$\begin{aligned} \|\Pi^+(t)v_\lambda(t, t_0) - \Pi^+(t)v_\lambda(t, r)\| &\leq \kappa e^{-\beta(t-r)}\widehat{C}(1, \gamma) \sup_{s \in [t_0, r]} e^{\gamma(r-s)}\|f(s)\| \\ &\leq \kappa e^{-\beta(t-r)}\widehat{C}(1, \gamma)\|f\|_\eta \sup_{s \in [t_0, r]} e^{\gamma(r-s)}e^{\eta|s|} \\ &\leq \kappa e^{-\beta(t-r)}\widehat{C}(1, \gamma)\|f\|_\eta e^{-\gamma(t-r)} \sup_{s \in [t_0, r]} e^{\gamma(t-s)}e^{\eta|s|} \\ &\leq \kappa e^{-(\beta+\gamma)(t-r)}\widehat{C}(1, \gamma)\|f\|_\eta \sup_{s \in [t_0, r]} e^{\gamma(t-s)}e^{\eta(|t-s|+|t|)}. \end{aligned}$$

Then using the fact that $\beta + \gamma > 0$ and $\eta + \gamma < 0$ we obtain

$$\|\Pi^+(t)v_\lambda(t, t_0) - \Pi^+(t)v_\lambda(t, r)\| \leq \kappa e^{-(\beta+\gamma)(t-r)}\widehat{C}(1, \gamma)\|f\|_\eta e^{\eta|t|},$$

that is

$$\lim_{t_0, r \rightarrow -\infty} \|\Pi^+(t)v_\lambda(t, t_0) - \Pi^+(t)v_\lambda(t, r)\| = 0.$$

This prove the existence of the limit (5.15) for any fixed $t \in \mathbb{R}$.

Proof of i): Let $\eta \in [0, \beta)$ be given. Let $\nu \in (-\beta, 0)$. By Proposition 5.8 we can find a constant $\widehat{C}(1, \nu) > 0$ such that

$$\|\Pi^+(t)v_\lambda(t, t_0)\| \leq \widehat{C}(1, \nu) \sup_{s \in [t_0, t]} e^{\nu(t-s)} \|f(s)\|, \quad \forall (t, t_0) \in \Delta.$$

Then for all $(t, t_0) \in \Delta$

$$\begin{aligned} \|\Pi^+(t)v_\lambda(t, t_0)\| &\leq \widehat{C}(1, \nu) \sup_{s \in [t_0, t]} e^{\nu(t-s)} \|f(s)\| \\ &\leq \widehat{C}(1, \nu) \|f\|_\eta \sup_{s \in [t_0, t]} e^{\nu(t-s)} e^{\eta|s|} \\ &\leq \widehat{C}(1, \nu) \|f\|_\eta \sup_{s \in [t_0, t]} e^{\nu(t-s)} e^{\eta|t-s| + \eta|t|} \\ &\leq \widehat{C}(1, \nu) \|f\|_\eta e^{\eta|t|} \sup_{s \in [t_0, t]} e^{(\nu+\eta)(t-s)}, \end{aligned}$$

and since $\nu + \eta < 0$ we obtain

$$\|\Pi^+(t)v_\lambda(t, t_0)\| \leq \widehat{C}(1, \nu) \|f\|_\eta e^{\eta|t|}. \quad (5.18)$$

The result follows by letting $t_0 \rightarrow -\infty$ in (5.18).

Proof of ii): Let $\eta \in [0, \beta)$ and $(t, l) \in \Delta$ be given. Then

$$\begin{aligned} \mathcal{J}_\lambda^+(f)(t) &= U_B^+(t, l) \int_{-\infty}^l U_B^+(l, s) \lambda R_\lambda(A) f(s) ds + \int_l^t U_B^+(t, s) \lambda R_\lambda(A) f(s) ds \\ &= U_B^+(t, l) \mathcal{J}_\lambda^+(f)(l) + \int_l^t U_B^+(t, s) \lambda R_\lambda(A) f(s) ds. \end{aligned}$$

■

Lemma 5.11 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Let $\eta \in [0, \beta)$ be given. Then for each $\lambda > \omega + 1$, each $f \in BC^\eta(\mathbb{R}, X)$ and $t_0 \in \mathbb{R}$

$$\mathcal{J}_\lambda^-(f)(t_0) := - \lim_{t \rightarrow +\infty} \int_{t_0}^t U_B^-(t_0, s) \lambda R_\lambda(A) f(s) ds := - \int_{t_0}^{+\infty} U_B^-(t_0, s) \lambda R_\lambda(A) f(s) ds, \quad (5.19)$$

exists. Moreover the following properties hold

- i) *For each $\eta \in [0, \beta)$ and each $\lambda > \omega + 1$, \mathcal{J}_λ^- is a bounded linear operator from $BC^\eta(\mathbb{R}, X)$ into itself. More precisely for any $\nu \in (-\beta, 0)$*

$$\|\mathcal{J}_\lambda^-(f)\|_\eta \leq \widehat{C}(1, \nu) \|f\|_\eta, \quad \forall f \in BC^\eta(\mathbb{R}, X) \quad \text{with } \eta \in [0, -\nu],$$

where $\widehat{C}(1, \nu)$ is the constant introduced in Proposition 5.8.

ii) For each $\eta \in [0, \beta)$, each $\lambda > \omega + 1$ and each $f \in BC^\eta(\mathbb{R}, X)$ we have

$$\mathcal{J}_\lambda^-(f)(t) = U_B^-(t, l)\mathcal{J}_\lambda^-(f)(l) + \int_l^t U_B^-(t, s)\lambda R_\lambda(A)f(s)ds, \quad \forall (t, l) \in \Delta. \quad (5.20)$$

Proof. Let $\eta \in [0, \beta)$ be given. Let $\lambda > \omega + 1$ be given and fixed. Recall

$$v_\lambda(t, t_0) := \int_{t_0}^t U_B(t, s)\lambda R_\lambda(A)f(s)ds, \quad \forall (t, t_0) \in \Delta.$$

Observe that

$$U_B^-(t_0, t)v_\lambda(t, t_0) = \int_{t_0}^t U_B^-(t_0, s)\lambda R_\lambda(A)f(s)ds, \quad \forall (t, t_0) \in \Delta,$$

and

$$\mathcal{J}_\lambda^-(f)(t_0) = -\lim_{t \rightarrow +\infty} z_\lambda(t, t_0), \quad \forall t_0 \in \mathbb{R},$$

with

$$z_\lambda(t, t_0) := U_B^-(t_0, t)v_\lambda(t, t_0), \quad \forall (t, t_0) \in \Delta. \quad (5.21)$$

To prove the existence of the limit, we will show that for each $t_0 \in \mathbb{R}$, $\{w_\lambda(t, t_0)\}_{t \geq t_0}$ is a Cauchy sequence. Let $f \in BC^\eta(\mathbb{R}, X)$ be given. Let $t, r \in \mathbb{R}$ such that $t_0 \leq r \leq t$. Then we have

$$\begin{aligned} z_\lambda(t, t_0) &= \int_{t_0}^r U_B^-(t_0, s)\lambda R_\lambda(A)f(s)ds + U_B^-(t_0, r) \int_r^t U_B^-(r, s)\lambda R_\lambda(A)f(s)ds \\ &= z_\lambda(r, t_0) + U_B^-(t_0, r)z_\lambda(r, t), \end{aligned}$$

and

$$z_\lambda(t, t_0) - z_\lambda(r, t_0) = U_B^-(t_0, r)z_\lambda(r, t). \quad (5.22)$$

Then by Proposition 5.9 and the definition of z_λ in (5.21) we can find a constant $\widehat{C}(1, \gamma) > 0$ with $\gamma \in (-\beta, -\eta)$ such that

$$\begin{aligned} \|z_\lambda(t, t_0) - z_\lambda(r, t_0)\| &\leq \kappa e^{-\beta(r-t_0)} \widehat{C}(1, \gamma) \sup_{s \in [t, r]} e^{\gamma(s-t)} \|f(s)\| \\ &\leq \kappa e^{-\beta(r-t_0)} \widehat{C}(1, \gamma) \|f\|_\eta e^{-\gamma(t-t_0)} \sup_{s \in [t, r]} e^{\gamma(s-t_0)} e^{\eta|s|} \\ &\leq \kappa e^{-\beta(r-t_0)} \widehat{C}(1, \gamma) \|f\|_\eta e^{-\gamma(r-t_0)} e^{-\gamma(t-r)} \sup_{s \in [t, r]} e^{\gamma(s-t_0)} e^{\eta|s|} \\ &\leq \kappa e^{-(\beta+\gamma)(r-t_0)} \widehat{C}(1, \gamma) \|f\|_\eta e^{-\gamma(t-r)} \sup_{s \in [t, r]} e^{\gamma(s-t_0)} e^{\eta|s|} \\ &\leq \kappa e^{-(\beta+\gamma)(r-t_0)} \widehat{C}(1, \gamma) \|f\|_\eta \sup_{s \in [t, r]} e^{\gamma(s-t_0)} e^{\eta|s|} \\ &\leq \kappa e^{-(\beta+\gamma)(r-t_0)} \widehat{C}(1, \gamma) \|f\|_\eta \sup_{s \in [t, r]} e^{\gamma(s-t_0)} e^{\eta(|s-t_0|+|t_0|)} \\ &\leq \kappa e^{-(\beta+\gamma)(r-t_0)} \widehat{C}(1, \gamma) \|f\|_\eta \sup_{s \in [t, r]} e^{(\gamma+\eta)(s-t_0)} e^{\eta|t_0|}, \end{aligned}$$

and since $\beta + \gamma > 0$, $\gamma + \eta < 0$ we obtain

$$\|z_\lambda(t, t_0) - z_\lambda(r, t_0)\| \leq \kappa e^{-(\beta+\gamma)(r-t_0)} \widehat{C}(1, \gamma) \|f\|_\eta e^{\eta|t_0|},$$

which gives

$$\lim_{t, r \rightarrow +\infty} \|z_\lambda(t, t_0) - z_\lambda(r, t_0)\| = 0,$$

and proves the existence of the limit (5.19).

Proof of i): Let $\eta \in [0, \beta)$ be given. Let $\nu \in (-\beta, 0)$. Note that by Proposition 5.9 we can find a constant $\widehat{C}(1, \nu) > 0$ such that

$$\|w_\lambda(t, t_0)\| \leq \widehat{C}(1, \nu) \sup_{s \in [t_0, t]} e^{\gamma(s-t_0)} \|f(s)\|, \quad \forall (t, t_0) \in \Delta.$$

Then for all $(t, t_0) \in \Delta$

$$\begin{aligned} \|w_\lambda(t, t_0)\| &\leq \widehat{C}(1, \nu) \|f\|_\eta \sup_{s \in [t_0, t]} e^{\nu(s-t_0)} e^{\eta|s|} \\ &\leq \widehat{C}(1, \nu) \|f\|_\eta \sup_{s \in [t_0, t]} e^{\nu(s-t_0)} e^{\eta(|s-t_0|+|t_0|)} \\ &\leq \widehat{C}(1, \nu) \|f\|_\eta \sup_{s \in [t_0, t]} e^{(\nu+\eta)(s-t_0)} e^{\eta|t_0|}, \end{aligned}$$

and since $\nu + \eta < 0$ we obtain

$$\|w_\lambda(t, t_0)\| \leq \widehat{C}(1, \nu) \|f\|_\eta e^{\eta|t_0|}. \quad (5.23)$$

The result follows by letting $t \rightarrow +\infty$ in (5.23).

Proof of ii): Let $\eta \in [0, \beta)$ and $(t, l) \in \Delta$ be given. Then

$$\begin{aligned} \mathcal{J}_\lambda^-(f)(l) &= - \int_l^t U_B^-(l, s) \lambda R_\lambda(A) f(s) ds - \int_t^{+\infty} U_B^-(l, s) \lambda R_\lambda(A) f(s) ds \\ &= -U_B^-(l, t) \int_l^t U_B^-(t, s) \lambda R_\lambda(A) f(s) ds - U_B^-(l, t) \int_t^{+\infty} U_B^-(t, s) \lambda R_\lambda(A) f(s) ds \end{aligned}$$

because $U_B^-(l, t)$ is invertible from $\Pi^-(t)(X_0)$ into $\Pi^-(l)(X_0)$ with inverse $U_B^-(t, l)$ and $\mathcal{J}_\lambda^-(f)(l) \in \Pi^-(l)(X_0)$ one gets

$$\begin{aligned} U_B^-(t, l) \mathcal{J}_\lambda^-(f)(l) &= - \int_l^t U_B^-(t, s) \lambda R_\lambda(A) f(s) ds - \int_t^{+\infty} U_B^-(t, s) \lambda R_\lambda(A) f(s) ds \\ &= - \int_l^t U_B^-(t, s) \lambda R_\lambda(A) f(s) ds + \mathcal{J}_\lambda^-(f)(t), \end{aligned}$$

and the result follows. ■

Lemma 5.12 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Let $\eta \in [0, \beta)$ be given. For each $\lambda > \omega + 1$ and each $f \in BC^\eta(\mathbb{R}, X)$ define

$$\mathcal{J}_\lambda(f)(t) := \mathcal{J}_\lambda^+(f)(t) + \mathcal{J}_\lambda^-(f)(t) := \int_{-\infty}^{+\infty} \Gamma_B(t, s) \lambda R_\lambda(A) f(s) ds, \quad \forall t \in \mathbb{R}, \quad (5.24)$$

Then the following properties hold

i) For each $\eta \in [0, \beta)$ and each $\lambda > \omega + 1$, \mathcal{J}_λ is a bounded linear operator from $BC^\eta(\mathbb{R}, X)$ into itself. More precisely for any $\nu \in (-\beta, 0)$

$$\|\mathcal{J}_\lambda(f)\|_\eta \leq 2 \widehat{C}(1, \nu) \|f\|_\eta, \quad \forall f \in BC^\eta(\mathbb{R}, X) \quad \text{with } \eta \in [0, -\nu], \quad (5.25)$$

where $\widehat{C}(1, \nu)$ is the constant introduced in Proposition 5.8.

ii) For each $\eta \in [0, \beta)$, each $\lambda > \omega + 1$ and each $f \in BC^\eta(\mathbb{R}, X)$ we have

$$\mathcal{J}_\lambda(f)(t) = U_B(t, l) \mathcal{J}_\lambda(f)(l) + \int_l^t U_B(t, s) \lambda R_\lambda(A) f(s) ds, \quad \forall (t, l) \in \Delta. \quad (5.26)$$

iii) For each $\eta \in [0, \beta)$, each $f \in BC^\eta(\mathbb{R}, X)$, $\mathcal{J}_\lambda(f)$ is uniformly convergent on compact subset of \mathbb{R} as $\lambda \rightarrow +\infty$.

iv) For each $f \in BUC(\mathbb{R}, X) \subset BC^0(\mathbb{R}, X)$ with relatively compact range, $\mathcal{J}_\lambda(f)$ is uniformly convergent on \mathbb{R} as $\lambda \rightarrow +\infty$.

Proof. The proof of i) follows from Lemmas 5.10 and 5.11.

Proof of ii) Let $\eta \in [0, \beta)$, $f \in BC^\eta(\mathbb{R}, X)$ and $(t, l) \in \Delta$ be given. Since $\mathcal{J}_\lambda^+(f)(l) \in \Pi^+(l)$, $\mathcal{J}_\lambda^-(f)(l) \in \Pi^-(l)$ one gets from (5.16) and (5.20)

$$\mathcal{J}_\lambda^+(f)(t) = U_B(t, l) \Pi^+(l) \mathcal{J}_\lambda^+(f)(l) + \Pi^+(t) \int_l^t U_B(t, s) \lambda R_\lambda(A) f(s) ds. \quad (5.27)$$

and

$$\mathcal{J}_\lambda^-(f)(t) = U_B(t, l) \Pi^-(l) \mathcal{J}_\lambda^-(f)(l) + \Pi^-(t) \int_l^t U_B(t, s) \lambda R_\lambda(A) f(s) ds. \quad (5.28)$$

and the result follows by adding up (5.27) and (5.28) combined with the fact that $\Pi^+(t) + \Pi^-(t) = \Pi^+(l) + \Pi^-(l) = I$.

Proof of iii) To do this we will prove the convergence for \mathcal{J}_λ^+ and \mathcal{J}_λ^- as λ goes to $+\infty$. Let $\eta \in [0, \beta)$ and $f \in BC^\eta(\mathbb{R}, X)$ be given.

Let $\varepsilon > 0$ be given and fixed. Let $r > 0$ be large enough such that

$$2\kappa e^{-\beta r} \widehat{C}(1, \nu) \|f\|_\eta \leq \varepsilon. \quad (5.29)$$

We first prove the convergence for \mathcal{J}_λ^+ as λ goes to $+\infty$. Indeed by using (5.27) combined with the estimate in *i)* we obtain for each $\lambda, \mu > \omega + 1$, each $t \in \mathbb{R}$

$$\begin{aligned} \|\mathcal{J}_\lambda^+(f)(t) - \mathcal{J}_\mu^+(f)(t)\| &\leq \kappa e^{-\beta r} \|\mathcal{J}_\lambda^+(f)(t-r) - \mathcal{J}_\mu^+(f)(t-r)\| \\ &\quad + \kappa \left\| \int_{t-r}^t U_B(t, s) [\lambda R_\lambda(A) - \mu R_\mu(A) f(s)] ds \right\| \\ &\leq 2\kappa e^{-\beta r} \widehat{C}(1, \nu) \|f\|_\eta \\ &\quad + \kappa \left\| \int_{t-r}^t U_B(t, s) [\lambda R_\lambda(A) - \mu R_\mu(A) f(s)] ds \right\| \end{aligned}$$

and by using (5.29) we obtain the estimate

$$\|\mathcal{J}_\lambda^+(f)(t) - \mathcal{J}_\mu^+(f)(t)\| \leq \varepsilon + \kappa \left\| \int_{t-r}^t U_B(t, s) [\lambda R_\lambda(A) - \mu R_\mu(A) f(s)] ds \right\|, \quad \forall t \in \mathbb{R}. \quad (5.30)$$

Now infer from Theorem 1.6 that

$$\lim_{\lambda, \mu \rightarrow +\infty} \int_{t-r}^t U_B(t, s) [\lambda R_\lambda(A) - \mu R_\mu(A) f(s)] ds = 0,$$

uniformly for t in a compact subset of \mathbb{R} and (5.30) yields

$$\lim_{\lambda, \mu \rightarrow +\infty} \|\mathcal{J}_\lambda^+(f)(t) - \mathcal{J}_\mu^+(f)(t)\| \leq \varepsilon,$$

uniformly for t in a compact subset of \mathbb{R} . Since $\varepsilon > 0$ is arbitrary fixed we conclude by a Cauchy sequence argument that $\lim_{\lambda \rightarrow +\infty} \mathcal{J}_\lambda^+(f)(t)$ exists uniformly for t in a compact subset of \mathbb{R} .

Now we prove the convergence for \mathcal{J}_λ^- . First recall that for each $t \in \mathbb{R}$, $U_B^-(t+r, t)$ is invertible from $\Pi^-(t)(X_0)$ into $\Pi^-(t+r)(X_0)$ with inverse $U_B^-(t, t+r)$. Then by applying $U_B^-(t, t+r)$ to the left side of (5.28) one gets for all $t \in \mathbb{R}$

$$\begin{aligned} U_B^-(t, t+r) \mathcal{J}_\lambda^-(f)(t+r) &= U_B^-(t, t+r) U_B(t+r, t) \Pi^-(t) \mathcal{J}_\lambda^-(f)(t) \\ &\quad + U_B^-(t, t+r) \Pi^-(t+r) \int_t^{t+r} U_B(t+r, s) \lambda R_\lambda(A) f(s) ds, \quad \forall t \in \mathbb{R}, \end{aligned}$$

that is

$$U_B^-(t, t+r) \mathcal{J}_\lambda^-(f)(t+r) = \mathcal{J}_\lambda^-(f)(t) + U_B^-(t, t+r) \int_t^{t+r} U_B(t+r, s) \lambda R_\lambda(A) f(s) ds, \quad \forall t \in \mathbb{R},$$

so that

$$\mathcal{J}_\lambda^-(f)(t) = U_B^-(t, t+r) \mathcal{J}_\lambda^-(f)(t+r) - U_B^-(t, t+r) \int_t^{t+r} U_B(t+r, s) \lambda R_\lambda(A) f(s) ds, \quad \forall t \in \mathbb{R}.$$

Then for each $\lambda, \mu > \omega + 1$, each $t \in \mathbb{R}$

$$\begin{aligned} \|\mathcal{J}_\lambda^-(f)(t) - \mathcal{J}_\mu^-(f)(t)\| &\leq \kappa e^{-\beta r} \|\mathcal{J}_\lambda^-(f)(t+r) - \mathcal{J}_\mu^-(f)(t+r)\| \\ &\quad + \left\| \int_t^{t+r} U_B^-(t, s) [\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds \right\| \\ &\leq 2\kappa e^{-\beta r} \widehat{C}(1, \nu) \|f\|_\eta \\ &\quad + \kappa \left\| \int_t^{t+r} U_B(t+r, s) [\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds \right\|, \end{aligned}$$

and by using (5.29) we obtain the estimate

$$\|\mathcal{J}_\lambda^-(f)(t) - \mathcal{J}_\mu^-(f)(t)\| \leq \varepsilon + \kappa \left\| \int_t^{t+r} U_B(t+r, s) [\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds \right\|. \quad (5.31)$$

Now we infer from Theorem 1.6 that

$$\lim_{\lambda, \mu \rightarrow +\infty} \int_t^{t+r} U_B(t+r, s) [\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds = 0,$$

uniformly for t in a compact subset of \mathbb{R} and (5.31) yields

$$\lim_{\lambda, \mu \rightarrow +\infty} \|\mathcal{J}_\lambda^-(f)(t) - \mathcal{J}_\mu^-(f)(t)\| \leq \varepsilon.$$

uniformly for t in a compact subset of \mathbb{R} . Since $\varepsilon > 0$ is arbitrary fixed we conclude by a Cauchy sequence argument that $\lim_{\lambda \rightarrow +\infty} \mathcal{J}_\lambda^-(f)(t)$ exists uniformly for t in a compact subset of \mathbb{R} .

Finally we obtain that

$$\lim_{\lambda \rightarrow +\infty} \mathcal{J}_\lambda(f)(t) = \lim_{\lambda \rightarrow +\infty} \mathcal{J}_\lambda^+(f)(t) + \lim_{\lambda \rightarrow +\infty} \mathcal{J}_\lambda^-(f)(t),$$

exists uniformly for t in a compact subset of \mathbb{R} .

Proof of iv) The proof use the same argument as in the proof of iii). The uniform convergence on \mathbb{R} is obtained by using Proposition 4.1 which ensures that the limits

$$\lim_{\lambda, \mu \rightarrow +\infty} \int_{t-r}^t U_B(t, s) [\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds = 0,$$

and

$$\lim_{\lambda, \mu \rightarrow +\infty} \int_t^{t+r} U_B(t+r, s) [\lambda R_\lambda(A) - \mu R_\mu(A)] f(s) ds = 0,$$

are uniform for $t \in \mathbb{R}$. ■

Now we are ready to prove the analogue of Theorem 5.4 for our purpose.

Theorem 5.13 *Let Assumptions 1.1, 1.2 and 1.3 be satisfied. Assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Then the following assertions are equivalent

- i) The evolution family $\{U_B(t, s)\}_{(t,s) \in \Delta}$ has an exponential dichotomy.
- ii) For each $f \in BC(\mathbb{R}, X)$, there exists a unique integrated solution $u \in BC(\mathbb{R}, X_0)$ of (1.1).

Moreover if U_B has an exponential dichotomy with exponent $\beta > 0$, then for each $\eta \in [0, \beta)$ and each $f \in BC^\eta(\mathbb{R}, X)$ there exists a unique integrated solution $u \in BC^\eta(\mathbb{R}, X_0)$ of (1.1) which is given by

$$u(t) = \lim_{\lambda \rightarrow +\infty} \mathcal{J}_\lambda(f)(t) = \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{+\infty} \Gamma_B(t, s) \lambda R_\lambda(A) f(s) ds, \quad \forall t \in \mathbb{R},$$

where $\{\Gamma_B(t, s)\}_{(t,s) \in \mathbb{R}^2}$ is the Green's operator function associated to $\{U_B(t, s)\}_{(t,s) \in \Delta}$.

Proof. *i) \Rightarrow ii)* This is a direct consequence of Lemma 5.12 by taking the limit when λ goes to $+\infty$ in (5.26).

ii) \Rightarrow i) First of all note that since $BC(\mathbb{R}, X_0) \subset BC(\mathbb{R}, X)$, the property ii) ensures that for each $f \in BC(\mathbb{R}, X_0)$ there exists a unique integrated solution $u \in BC(\mathbb{R}, X_0)$ of (1.1). Furthermore note that if $u_f \in BC(\mathbb{R}, X_0)$ is a solution of (1.1) for $f \in BC(\mathbb{R}, X_0)$ then by Corollary 3.1 we know that it satisfies the integral equation

$$u_f(t) = U_B(t, t_0)x_0 + \int_{t_0}^t U_B(t, s)f(s)ds, \quad \forall t \geq t_0,$$

and i) follows by Theorem 5.4. The proof is complete. ■

As a consequence of the foregoing theorem we can obtain the following persistence result for exponential dichotomy

Theorem 5.14 *Let Assumptions 1.1, 1.2, 1.3 and 5.6 be satisfied and assume in addition that*

$$\sup_{t \in \mathbb{R}} b(t) < +\infty.$$

Then there exists $\varepsilon > 0$ such that for each strongly continuous family $\{C(t)\}_{t \in \mathbb{R}} \subset \mathcal{L}(X_0, X)$ satisfying

$$\sup_{t \in \mathbb{R}} \|B(t) - C(t)\|_{\mathcal{L}(X_0, X)} \leq \varepsilon,$$

the evolution family generated by

$$\frac{du(t)}{dt} = (A + C(t))u(t), \quad \text{for } t \in \mathbb{R}. \quad (5.32)$$

has an exponential dichotomy.

Proof. The proof of this theorem is classical. Then we will only sketch the proof. Note that the evolution family generated by (5.32) has an exponential dichotomy if and only if for each $f \in BC(\mathbb{R}, X)$ there exists a unique $u \in BC(\mathbb{R}, X_0)$ satisfying

$$\frac{du(t)}{dt} = (A + C(t))u(t) + f(t), \quad \text{for } t \in \mathbb{R}.$$

or equivalently

$$\frac{du(t)}{dt} = (A + B(t))u(t) + [C(t) - B(t)]u(t) + f(t), \text{ for } t \in \mathbb{R}.$$

This is equivalent to solve for each $f \in BC(\mathbb{R}, X)$ the fixed point problem to find $u \in BC(\mathbb{R}, X_0)$

$$u(t) = \mathcal{J}([C(\cdot) - B(\cdot)]u(\cdot) + f(\cdot))(t)$$

where

$$\mathcal{J}(g)(t) = \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{+\infty} \Gamma_B(t, s) \lambda R_\lambda(A) g(s) ds, \quad \forall t \in \mathbb{R}$$

which can be performed by using the uniform estimates (5.25) (for $\eta = 0$) obtained in Lemma 5.12. ■

6 Example

In order to illustrate our results we will apply some of the result to parabolic equation. Let $p \in [1, +\infty)$ and $I := (0, 1)$. Consider the following parabolic equation with non local boundary condition for each initial time $t_0 \in \mathbb{R}$

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + \alpha u(t, x) + g(t, x), \text{ for } t \geq t_0 \text{ and } x \in (0, 1), \\ -\frac{\partial u(t, 0)}{\partial x} = \int_I \beta_0(t, x) \varphi(x) dx + h_0(t) \\ +\frac{\partial u(t, 1)}{\partial x} = \int_I \beta_1(t, x) \varphi(x) dx + h_1(t) \\ u(t_0, \cdot) = \varphi \in L^p(I, \mathbb{R}), \end{cases} \quad (6.1)$$

with $\alpha > 0$, $g \in C(\mathbb{R}, L^p(I, \mathbb{R}))$, $h_0, h_1 \in C(\mathbb{R}, \mathbb{R})$ and $\beta_0, \beta_1 \in C(\mathbb{R}, L^q(I, \mathbb{R}))$ (with $\frac{1}{p} + \frac{1}{q} = 1$).

Abstract reformulation: In order to incorporate the boundary condition into the state variable, we consider

$$X := \mathbb{R}^2 \times L^p(I, \mathbb{R})$$

which is a Banach space endowed with the usual product norm

$$\left\| \begin{pmatrix} x_0 \\ x_1 \\ \varphi \end{pmatrix} \right\| = |x_0| + |x_1| + \|\varphi\|_{L^p}$$

and we set

$$X_0 := \{0_{\mathbb{R}^2}\} \times L^p(I, \mathbb{R}).$$

We consider $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ the linear operator defined by

$$\mathcal{A} \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} := \begin{pmatrix} \varphi'(0) \\ -\varphi'(1) \\ \varphi'' \end{pmatrix}$$

with

$$D(\mathcal{A}) := \{0_{\mathbb{R}^2}\} \times W^{2,p}(I, \mathbb{R}).$$

By construction \mathcal{A}_0 the part of \mathcal{A} in X_0 coincides with the usual formulation for the parabolic equation (6.1) with homogeneous boundary conditions. Indeed $\mathcal{A}_0 : D(\mathcal{A}_0) \subset X_0 \rightarrow X_0$ is a linear operator on X_0 defined by

$$\mathcal{A}_0 \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi'' \end{pmatrix}$$

with

$$D(\mathcal{A}_0) = \{0_{\mathbb{R}^2}\} \times \{\varphi \in W^{2,p}((0,1), \mathbb{R}) : \varphi'(0) = \varphi'(1) = 0\}.$$

In the following lemma we will first summarize some classical properties for the linear operator \mathcal{A}_0 .

Lemma 6.1 *The linear operator \mathcal{A}_0 is the infinitesimal generator of $\{T_{\mathcal{A}_0}(t)\}_{t \geq 0}$ an analytic semigroup of bounded linear operator on X_0 . Moreover $T_{\mathcal{A}_0}(t)$ is compact for each $t > 0$ and $(0, +\infty) \subset \rho(\mathcal{A}_0)$. The spectrum of \mathcal{A}_0 is given by*

$$\sigma(\mathcal{A}_0) = \{-(\pi k)^2 : k \in \mathbb{N}\}$$

and each eigenvalue $\lambda_k := -(\pi k)^2$ is associated to the eigenfunction

$$\psi_k(x) := \sin(\pi k x).$$

Moreover each eigenvalue λ_k is simple and the projector on the generalized eigenspace associated to this eigenvalue is given by

$$\Pi_{k,0} \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} := \begin{pmatrix} 0_{\mathbb{R}^2} \\ \frac{\int_0^1 \psi_k(r) \varphi(r) dr}{\int_0^1 \psi_k(r)^2 dr} \psi_k \end{pmatrix}.$$

Set

$$\Omega_\omega = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > \omega\}, \quad \forall \omega \in \mathbb{R},$$

define for $\lambda \in \mathbb{C}$,

$$\Delta(\lambda) := \mu^2(e^\mu - e^{-\mu}),$$

where

$$\mu := \sqrt{\lambda}.$$

Next we compute explicitly the resolvent of \mathcal{A} .

Lemma 6.2 *The resolvent of \mathcal{A} is given by For each $\omega_{\mathcal{A}} \geq 0$, such that*

$$\Omega_{\omega_{\mathcal{A}}} \subset \{\lambda \in \mathbb{C} : \Delta(\lambda) \neq 0\} \subset \rho(\mathcal{A}),$$

and for each $\lambda := \mu^2 \in \Omega_{\omega_{\mathcal{A}}}$ we have

$$\begin{aligned} (\lambda I - \mathcal{A}) \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} &= \begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix} \Leftrightarrow \\ \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix} &= (\lambda I - \mathcal{A})^{-1} \begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix} \Leftrightarrow \\ \varphi(x) &= \frac{\Delta_1(x)}{\Delta(\lambda)} \frac{1}{\mu} y_0 + \frac{\Delta_2(x)}{\Delta(\lambda)} \frac{1}{\mu} y_1 + \frac{\Delta_1(x)}{\Delta(\lambda)} \frac{1}{2\mu} \int_0^1 e^{-\mu s} f(s) ds \\ &\quad + \frac{\Delta_2(x)}{\Delta(\lambda)} \frac{1}{2\mu} \int_0^1 e^{-\mu(1-s)} f(s) ds + \frac{1}{2\mu} \int_0^1 e^{-\mu|x-s|} f(s) ds \end{aligned}$$

where

$$\Delta_1(x) = \mu^2 [e^{\mu(1-x)} + e^{-\mu(1-x)}] \text{ and } \Delta_2(x) = \mu^2 [e^{-\mu x} + e^{\mu x}].$$

Proof. In order to compute the resolvent we set

$$u(x) := \frac{1}{2\mu} \int_0^1 e^{-\mu|x-s|} f(s) ds = \frac{1}{2\mu} \int_{-\infty}^{+\infty} e^{-\mu|x-s|} \bar{f}(s) ds$$

where \bar{f} extend f by 0 on $\mathbb{R} \setminus [0, 1]$. We have

$$u(x) = \frac{1}{2\mu} \left[\int_{-\infty}^x e^{-\mu(x-s)} \bar{f}(s) ds + \int_x^{+\infty} e^{\mu(x-s)} \bar{f}(s) ds \right]$$

so

$$u'(x) = -\frac{1}{2} \int_{-\infty}^x e^{-\mu(x-s)} \bar{f}(s) ds + \frac{1}{2} \int_x^{+\infty} e^{\mu(x-s)} \bar{f}(s) ds.$$

We set

$$u(0) = \gamma_0 := \frac{1}{2\mu} \int_0^1 e^{-\mu s} f(s) ds \text{ and } u(1) = \gamma_1 := \frac{1}{2\mu} \int_0^1 e^{-\mu(1-s)} f(s) ds$$

and we observe that

$$u'(0) = \mu\gamma_0 \text{ and } u'(1) = -\mu\gamma_1$$

We set

$$u_1(x) := e^{-\mu x} \text{ and } u_2(x) := e^{\mu x}.$$

In order to solve the problem

$$(\lambda I - \mathcal{A}) \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix}$$

we look for φ under the form

$$\varphi(x) = u(x) + z_1 u_1(x) + z_2 u_2(x),$$

where $z_1, z_2 \in \mathbb{R}$.

We observe that to verify the boundary conditions

$$-\varphi'(0) = y_0 \text{ and } \varphi'(1) = y_1$$

is equivalent to

$$\begin{cases} u'(0) + z_1 u_1'(0) + z_2 u_2'(0) = -y_0 \\ u'(0) + z_1 u_1'(0) + z_2 u_2'(0) = y_1 \end{cases}$$

so we must solve the system

$$\begin{aligned} z_1 u_1'(0) + z_2 u_2'(0) &= -y_0 - u'(0) \\ z_1 u_1'(1) + z_2 u_2'(1) &= y_1 - u'(1) \end{aligned}$$

which is equivalent to

$$\begin{aligned} -\mu z_1 + \mu z_2 &= -y_0 - \mu \gamma_0 \\ -\mu e^{-\mu} z_1 + \mu e^{\mu} z_2 &= y_1 + \mu \gamma_1 \end{aligned}$$

hence

$$\begin{aligned} z_1 &= \frac{1}{\Delta(\lambda)} [-\mu e^{\mu} (-y_0 - \mu \gamma_0) + \mu (y_1 + \mu \gamma_1)] \\ z_2 &= \frac{1}{\Delta(\lambda)} [-\mu e^{-\mu} (-y_0 - \mu \gamma_0) + \mu (y_1 + \mu \gamma_1)] \end{aligned}$$

and the result follows. ■

The following estimation shows that \mathcal{A} is not Hille-Yosida.

Lemma 6.3 *We have the following estimations*

$$0 < \liminf_{\lambda(\in \mathbb{R}) \rightarrow +\infty} \lambda^{\frac{1}{p^*}} \left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(X)} \leq \limsup_{\lambda(\in \mathbb{R}) \rightarrow +\infty} \lambda^{\frac{1}{p^*}} \left\| (\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(X)} < +\infty,$$

with

$$p^* = \frac{2p}{1+p}.$$

Proof. Let $\lambda > 0$ be large enough. We have

$$\left\| (\lambda I - \mathcal{A})^{-1} \begin{pmatrix} 0 \\ y_1 \\ 0_{L^p} \end{pmatrix} \right\| = |y_1| \frac{\sqrt{\lambda}}{\Delta(\lambda)} \left\| e^{\sqrt{\lambda} \cdot} + e^{-\sqrt{\lambda} \cdot} \right\|_{L^p}$$

Set

$$\gamma_\lambda := \frac{\sqrt{\lambda}}{\Delta(\lambda)} \left\| e^{\sqrt{\lambda} \cdot} + e^{-\sqrt{\lambda} \cdot} \right\|_{L^p}$$

we have

$$\frac{\sqrt{\lambda}}{\Delta(\lambda)} \left[\|e^{\sqrt{\lambda} \cdot}\|_{L^p} - \|e^{-\sqrt{\lambda} \cdot}\|_{L^p} \right] \leq \gamma_\lambda \leq \frac{\sqrt{\lambda}}{\Delta(\lambda)} \left[\|e^{\sqrt{\lambda} \cdot}\|_{L^p} + \|e^{-\sqrt{\lambda} \cdot}\|_{L^p} \right]$$

and

$$\frac{\sqrt{\lambda}}{\Delta(\lambda)} \|e^{\sqrt{\lambda} \cdot}\|_{L^p} = \frac{\sqrt{\lambda}}{\lambda(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})} \left(\int_0^1 e^{p\sqrt{\lambda}x} dx \right)^{1/p}$$

and

$$\lim_{\lambda \rightarrow +\infty} \gamma_\lambda \lambda^{\frac{p+1}{2p}} = (1/p)^{1/p} > 0,$$

and the result follows. ■

By using Lemmas 6.1-6.3, we deduce that Assumption 3.4 in Ducrot, Magal and Prevost [15] is satisfied. Therefore by applying Theorem 3.11 in [15] we obtain the following lemma.

Lemma 6.4 *The linear operator \mathcal{A} satisfies Assumption 1.1 and Assumption 1.2.*

Remark 6.5 *Since $\rho(\mathcal{A}) \neq \emptyset$, one can prove that $\sigma(\mathcal{A}_0) = \sigma(\mathcal{A})$ (see [31]).*

Abstract Cauchy problem: By identifying $u(t, \cdot)$ and $v(t) := \begin{pmatrix} 0_{\mathbb{R}^2} \\ u(t, \cdot) \end{pmatrix}$ we can rewrite equation (6.1) as the following abstract Cauchy problem for each initial time $t_0 \in \mathbb{R}$

$$\frac{dv(t)}{dt} = \mathcal{A}v(t) + \alpha v(t) + \mathcal{B}(t)v(t) + f(t), \text{ for } t \geq t_0 \text{ and } v(t_0) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix}, \quad (6.2)$$

where

$$\mathcal{B}(t) \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} := \begin{pmatrix} \int_I \beta_0(t, x) \varphi(x) dx \\ \int_I \beta_1(t, x) \varphi(x) dx \\ 0_{L^p} \end{pmatrix} \text{ and } f(t) := \begin{pmatrix} h_0(t) \\ h_1(t) \\ g(t, \cdot) \end{pmatrix}.$$

By using Lemma 6.1 we know that $(\mathcal{A} + \alpha I)_0$ the part of $(\mathcal{A} + \alpha I)$ is the infinitesimal generator of $\{T_{(\mathcal{A} + \alpha I)_0}(t)\}_{t \geq 0}$ an analytic semigroup of bounded linear on X_0 . By using Lemma 6.4 we deduce that $(\mathcal{A} + \alpha I)$ generates an integrated semigroup $\{S_{(\mathcal{A} + \alpha I)}(t)\}_{t \geq 0}$. Consider for each initial time $t_0 \in \mathbb{R}$ the parabolic equation

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + \alpha u(t, x), \text{ for } t \geq t_0 \text{ and } x \in (0, 1), \\ -\frac{\partial u(t, 0)}{\partial x} = \int_I \beta_0(t, x) \varphi(x) dx \\ +\frac{\partial u(t, 1)}{\partial x} = \int_I \beta_1(t, x) \varphi(x) dx \\ u(t_0, \cdot) = \varphi \in L^p(I, \mathbb{R}), \end{cases} \quad (6.3)$$

this equation corresponds to the abstract Cauchy problem for each initial time $t_0 \in \mathbb{R}$

$$\frac{dv(t)}{dt} = (\mathcal{A} + \alpha I)v(t) + \mathcal{B}(t)v(t), \text{ for } t \geq t_0 \text{ and } v(t_0) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ \varphi \end{pmatrix}. \quad (6.4)$$

Variation of constants formula : By using Proposition 1.5 we obtain the following result.

Proposition 6.6 *The Cauchy problem (6.4) generates a unique evolution family $\{U_{\mathcal{B}}(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(X_0)$. Moreover $U_{\mathcal{B}}(\cdot, t_0)x_0 \in C([t_0, +\infty), X_0)$ is the unique solution of the fixed point problem*

$$U_{\mathcal{B}}(t, t_0)x_0 = T_{(\mathcal{A} + \alpha I)_0}(t - t_0)x_0 + \frac{d}{dt} \int_{t_0}^t S_{(\mathcal{A} + \alpha I)}(t - s)\mathcal{B}(s)U_{\mathcal{B}}(s, t_0)x_0 ds, \quad t \geq t_0. \quad (6.5)$$

If we assume in addition that

$$\sup_{t \in \mathbb{R}} \|\beta_0(t, \cdot)\|_{L^q} + \|\beta_1(t, \cdot)\|_{L^q} < +\infty$$

then the evolution family $\{U_{\mathcal{B}}(t, s)\}_{(t,s) \in \Delta}$ is exponentially bounded.

By using Theorem 1.6 we obtain.

Theorem 6.7 *For each $t_0 \in \mathbb{R}$, each $x_0 \in X_0$ and each $f \in C([t_0, +\infty], X)$ the unique integrated solution $v_f \in C([t_0, +\infty], X_0)$ of (6.2) is given for each $t \geq t_0$ by*

$$v_f(t) = U_{\mathcal{B}}(t, t_0)x_0 + \lim_{\lambda \rightarrow +\infty} \int_{t_0}^t U_{\mathcal{B}}(t, s)\lambda R_{\lambda}(\mathcal{A} + \alpha I)f(s)ds \quad (6.6)$$

where the limit exists in X_0 . Moreover the convergence in (6.6) is uniform with respect to $t, t_0 \in I$ for each compact interval $I \subset \mathbb{R}$.

Exponential dichotomy result : By using Theorem 5.13 we obtain the following result

Theorem 6.8 *Assume that*

$$\sup_{t \in \mathbb{R}} \|\beta_0(t, \cdot)\|_{L^q} + \|\beta_1(t, \cdot)\|_{L^q} < +\infty$$

Then the following assertions are equivalent

- i) *The evolution family $\{U_{\mathcal{B}}(t, s)\}_{(t,s) \in \Delta}$ has an exponential dichotomy.*
- ii) *For each $f \in BC(\mathbb{R}, X)$, there exists a unique integrated solution $u \in BC(\mathbb{R}, X_0)$ of (6.2).*

Assumption 6.9 *Assume that $\alpha > 0$ and $\alpha \neq -(\pi k)^2, \forall k \in \mathbb{N}$.*

Then the spectrum of $\mathcal{A} + \alpha I$ do not contain any purely imaginary eigenvalue, and by using Lemma 6.1 and Remark 6.5 we deduce that

$$\sigma(\mathcal{A} + \alpha I) = \sigma(\mathcal{A}_0 + \alpha I) = \{-(\pi k)^2 + \alpha : k \in \mathbb{N}\}$$

therefore

$$0 \notin \sigma(\mathcal{A}_0 + \alpha I).$$

Then $U(t, s) := T_{\mathcal{A} + \alpha I}(t - s)$ has an exponential dichotomy and we can apply Theorem 5.14 with $A + B(t) := \mathcal{A} + \alpha I$ and $C(t) := \mathcal{B}(t)$.

Theorem 6.10 *Let Assumption 6.9 be satisfied. There exists $\varepsilon > 0$ such that*

$$\sup_{t \in \mathbb{R}} \|\beta_0(t, \cdot)\|_{L^q} + \|\beta_1(t, \cdot)\|_{L^q} < \varepsilon$$

implies that the evolution family $\{U_{\mathcal{B}}(t, s)\}_{(t,s) \in \Delta} \subset \mathcal{L}(X_0)$ has an exponential dichotomy.

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